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Invariant connections on homogeneous spaces

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Theorem (Wang, 1958).

Let K be a connected Lie group acting fibre-transitively as a group of automorphisms on a G-principal bundle P over a homogeneous space K/H and let H be the isotropy subgroup of $x_0 = \pi(p_0)$.
Then, there is a one-to-one correspondence between the set of K-invariant connections ω on P and the set of linear maps $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$ satisfying

1. $\Lambda(X) = \lambda_*(X), \quad X \in \mathfrak{h};$
2. $\Lambda \circ \mathrm{Ad}_H(h) = \mathrm{Ad}_G(\lambda(h)) \circ \Lambda, \quad h \in H,$

where $\lambda\colon H \to G$ is the Lie group homomorphism defined by

$$hp_0 = p_0\lambda(h), \quad h \in H.$$

The correspondence is given by

$$\Lambda(X) = \omega_{p_0}(\widehat{X}_{p_0}), \quad X \in \mathfrak{k},$$

where $\widehat{X} \in \mathfrak{X}(P)$ is the fundamental vector field on P corresponding to $X \in \mathfrak{k}$.

This is one of the most important theorems in the differential geometry of homogeneous spaces with additional G-structures.

In the present formulation, it's utterly incomprehensible.

In this talk:

- Explanation of context and typical examples
- Philosophical considerations: What is a connection?!?
- A peek at the proof
- The principal G-bundle dictionary
- Applications and outlook

Remarks: I am mainly interested in

- homogeneous reductive, but non-symmetric spaces,
- an additional semi-Riemannian metric
- an additional G-structure

Principal G-bundles and actions on them

Definition

Let P, M be manifolds, G a Lie group. A fiber bundle $P \xrightarrow{\pi} M$ is called a principal G-bundle if:

1. *The Lie group G acts from the right on the total space P,*
2. *The action of G on the fibers $P_x := \pi^{-1}(x)$ $(x \in M)$ is free and transitive along the fibers and preserves them, i. e. the action restricts to*

$$P_x \times G \to P_x,$$

   and the orbit map $a \mapsto pa$ is a bijection for all $p \in P_x,\ a \in G$,
3. *For every $x_0 \in M$ the local diffeomorphism*

$$\phi_{x_0} : \pi^{-1}\left(U_{x_0}\right) \to U_{x_0} \times G,$$

   is G-equivariant.

Symbolically:
$$\begin{tikzcd} P \arrow[d, "\pi"'] & G \arrow[l, "R_a"'] \\ M & \end{tikzcd}$$
for $a \in G$, R_a the right action

Examples. Frame bundle of M^n with $\mathrm{GL}(n,\mathbb{R})$-action, reductions, Hopf fibrations...

Let K/H be a homogeneous space. Then, $(K, \pi, K/H; H)$ is a principal H-bundle, r_h the right action of $h \in H$ on K:

$$\begin{tikzcd} K \arrow[d, "\pi"'] & H \arrow[l, "r_h"'] \\ K/H & \end{tikzcd}$$

Quaternionic spheres:
For $L = \{e\}, \mathrm{U}(1), \mathrm{Sp}(1)$:

$$\begin{tikzcd} \mathrm{Sp}(n+1) \cdot L \arrow[d, "\pi"'] & \mathrm{Sp}(n) \cdot L \arrow[l] \\ \mathbb{S}^{4n+3} & \end{tikzcd}$$

Stiefel manifolds:
For $G_{\mathbb{K}}(n) := \mathrm{O}(n),\ \mathrm{U}(n)$ resp. $\mathrm{Sp}(n)$:

$$\begin{tikzcd} G_{\mathbb{K}}(n) \arrow[d, "\pi"'] & G_{\mathbb{K}}(n-k) \arrow[l] \\ V_k(\mathbb{K}^n) & \end{tikzcd}$$

Algebraic approach to homogeneous spaces I

Endow M^n with a Riemannian metric g [M oriented].

A "classical" connection is a covariant derivative ∇ on TM. It's called *metric* if $\nabla g = 0$. This is a desirable property!

Important tensor fields:

- Curvature: $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$
- Torsion: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

Fact. $\exists_1$ metric torsion-free connection, called the Levi-Civita connection ∇^g.

Curvature and torsion capture different properties:

- Curvature measures deviation from a 'model space', i. e. "flat space".
- Torsion is a property of G-structures, if the connection happens to have a G-invariance property.

In general (Cartan, 1925) for $n \geq 5$:

$$T \in TM \otimes \bigwedge^2 TM \overset{\mathrm{SO}(n)}{\cong} TM \oplus \bigwedge^3 TM \oplus \mathcal{C} \quad \text{[we won't need } \mathcal{C}\text{]}$$

Ambrose-Singer Homogeneity Theorem, 1958.

A complete, connected and simply connected Riemannian manifold (M, g) is homogeneous if and only if it admits a covariant derivative ∇ satisfying

$$\nabla T = 0, \quad \nabla R = 0, \quad \nabla g = 0,$$

If $T = 0$ (hence $\nabla = \nabla^g$), this IS the geometric definition of a Riemannian symmetric space!

$\rightsquigarrow$ torsion measures deviation from a symm. space $\sim$ 'trivial' $\mathrm{SO}(n)$-structure

Idea: Distinguish homogeneous spaces by the irrep. in which its torsion lies!

Definition.

- The connection ∇ of the theorem is called the Ambrose-Singer connection;
- If T is in $\bigwedge^3 TM$, $M = K/H$ is called naturally reductive.

This is still not much structure. . .

Example: Homogeneous 3-(α, δ)-Sasaki manifolds

They are a generalisations of 3-Sasaki manifolds for some $\alpha \in \mathbb{R}^*, \ \delta \in \mathbb{R}$.

For us: Any homogeneous 3-(α, δ)-Sasaki manifold admits a canonical Riemannian submersion

$$P = K/L \longrightarrow M = P/\mathcal{F}_{\mathcal{V}}$$

over a homogeneous quaternionic Kähler space.

We assume $\delta \neq 0$ (non-degenerate case), so N has scalar curvature $s \neq 0$.

There are two known classes of homogeneous qK spaces:

- Compact qK symmetric spaces (Wolf spaces, $s > 0$) and their non-compact duals ($s < 0$);
- Alekseevsky-Cortes spaces, with $s < 0$ and admitting a transitive solvable group of isometries.

Intersection $\rightsquigarrow$ symmetric Alekseevsky spaces

Theorem (Boyer, Galicki, Mann, 1994).

Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a homogeneous 3-Sasaki manifold. Then M is one of the following homogeneous spaces:

$$\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}, \quad \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}, \quad \frac{\mathrm{SU}(m+2)}{S(\mathrm{U}(m)\times\mathrm{U}(1))}, \quad \frac{\mathrm{SO}(k+4)}{\mathrm{SO}(k)\times\mathrm{Sp}(1)},$$
$$\frac{\mathrm{G}_2}{\mathrm{Sp}(1)}, \quad \frac{\mathrm{F}_4}{\mathrm{Sp}(3)}, \quad \frac{\mathrm{E}_6}{\mathrm{SU}(6)}, \quad \frac{\mathrm{E}_7}{\mathrm{Spin}(12)}, \quad \frac{\mathrm{E}_8}{\mathrm{E}_7}.$$

Here $n \geq 0$, $m \geq 1$ and $k \geq 3$.

- They are all simply connected except for $\mathbb{R}P^{4n+3} \simeq \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}$
- 1-1 correspondence between simply connected 3-Sasaki homogeneous manifolds and compact simple Lie algebras

A uniform description of simply connected homogeneous models in Draper, Ortega, Palomo (2018) via the notion of a 3-Sasakian data

$\rightsquigarrow$ we generalized it

Definition

A generalized 3-Sasakian data is a triple (K, H, L) such that

- K *is a real simple Lie Group*
  (assumed compact simple in the non-generalized setting)
- $L \subset H \subset K$ *connected Lie subgroups*

and the Lie algebras $\mathfrak{l} \subset \mathfrak{h} \subset \mathfrak{k}$ satisfy:

- $(\mathfrak{k}, \mathfrak{h})$ *form a symmetric pair,* $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}$*,*
- $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{sp}(1)$ *with* $\mathfrak{sp}(1)$ *and* $\mathfrak{l}$ *commuting subalgebras,*
- *the complexification* $\mathfrak{p}^{\mathbb{C}} = \mathbb{C}^2 \otimes_{\mathbb{C}} W$ *for some* $\mathfrak{l}^{\mathbb{C}}$*-module of* $\dim_{\mathbb{C}} W = 2n$*,*
- $\mathfrak{l}^{\mathbb{C}}, \mathfrak{sp}(1)^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}}$ *act on* $\mathfrak{p}^{\mathbb{C}}$ *by their action on* W *and* $\mathbb{C}^2$*.*

This model contains compact symmetric pairs $(\mathfrak{k}, \mathfrak{h})$ and their non-compact symmetric duals $(\mathfrak{k}^*, \mathfrak{h})$.

$$\mathfrak{k} = \overbrace{\mathfrak{l} \oplus \underbrace{\mathfrak{sp}(1)}_{}}^{\mathfrak{h}}\!\phantom{\mathfrak{l} \oplus}\underbrace{\phantom{\mathfrak{sp}(1)} \oplus \mathfrak{p}}_{\mathfrak{m}} \qquad (\mathfrak{m} \text{ is a reductive complement for } P = K/L)$$

Theorem

Let (K, H, L) be generalized 3-Sasakian data. Let $\alpha, \delta \in \mathbb{R}$.

Assume $\alpha\delta > 0$ if K is compact, $\alpha\delta < 0$ if K is non-compact.

On $P = K/L$ define a K-invariant structure by the $\mathrm{Ad}(L)$-invariant tensors on $\mathfrak{m}$:

$$g\big|_{\mathfrak{sp}(1)} = \frac{-\kappa}{4\delta^2(n+2)}, \qquad g\big|_{\mathfrak{p}} = \frac{-\kappa}{8\alpha\delta(n+2)}, \qquad g\big|_{\mathfrak{sp}(1)\times\mathfrak{p}} = 0$$

$$\xi_i = \delta\sigma_i, \quad \eta_i = g(\xi_i, \cdot), \quad \varphi_i\big|_{\mathfrak{sp}(1)} = \frac{1}{2\delta}\,\mathrm{ad}(\xi_i), \quad \varphi_i\big|_{\mathfrak{p}} = \frac{1}{\delta}\,\mathrm{ad}(\xi_i).$$

κ Killing form on K, σ_i, $i = 1, 2, 3$ standard basis of $\mathfrak{sp}(1) \subset \mathfrak{h}$.

Then $(P, \varphi_i, \xi_i, \eta_i, g)$ defines a homogeneous 3-(α, δ)-Sasaki manifold, fibering over the symmetric qK space K/H.

Conversely every simply connected homogeneous 3-(α, δ)-Sasaki manifold fibering over a symmetric qK space is obtained by this construction.

- $(P/L, g)$ is naturally reductive $\Leftrightarrow \delta = 2\alpha \Leftrightarrow$ 'parallel' 3-(α, δ)-Sasaki manifolds.

In total, we obtain homogeneous 3-(α, δ)-Sasakian structures on the following list of homogeneous spaces (K/L compact, K^*/L non-compact):

K	K^*	L	H	dim
$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$4n+3$
$\mathrm{SU}(n+2)$	$\mathrm{SU}(n,2)$	$S(\mathrm{U}(n)\times\mathrm{U}(1))$	$S(\mathrm{U}(n)\mathrm{U}(2))$	$4n+3$
$\mathrm{SO}(n+4)$	$\mathrm{SO}(n,4)$	$\mathrm{SO}(n)\times\mathrm{Sp}(1)$	$\mathrm{SO}(n)\mathrm{SO}(4)$	$4n+3$
G_2	G_2^2	$\mathrm{Sp}(1)$	$\mathrm{SO}(4)$	11
F_4	F_4^{-20}	$\mathrm{Sp}(3)$	$\mathrm{Sp}(3)\mathrm{Sp}(1)$	31
E_6	E_6^2	$\mathrm{SU}(6)$	$\mathrm{SU}(6)\mathrm{Sp}(1)$	43
E_7	E_7^{-5}	$\mathrm{Spin}(12)$	$\mathrm{Spin}(12)\mathrm{Sp}(1)$	67
E_8	E_8^{-24}	E_7	$\mathrm{E}_7\mathrm{Sp}(1)$	115

.

What is the role of $\mathfrak{sp}(1)$ in alle these examples?

P happens to be a principal G-bundle for suitable G with $\mathfrak{g} = \mathfrak{sp}(1)$!

$K/L \to K/H$ has fibers isomorphic to G

ABSTRACT SETTING: PRINCIPAL G-BUNDLES WITH AN ADDITIONAL K-ACTION

Data: G-principal bundle $P \to M$, together with a Lie group K acting from the left on P s.t. the induced diffeomorphisms L_k of P are G-equivariant diffeomorphisms of P for all $k \in K \rightsquigarrow$ left action of K on M.

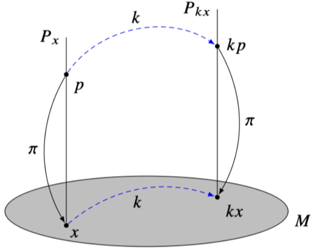


Figure 7.1. Induced left action of K on the base space M.

If the induced K-action on M is transitive, M is diffeomorphic to some K/H, where H the isotropy subgroup of any $x \in M$.

Two different group actions on a fixed fiber:

1. the isotropy subgroup H acts *from the left* on the fiber P_x over x,
2. the structure group G acts *from the right* on P_x.

Principal G-bundles with an additional K-action II

Q: Can these two actions on P_x be related to each other?

Definition.

We say that K acts fiber-transitively on the principal G-bundle $P \to M$ if

1. K acts on P as a group of automorphisms, and
2. for any two points $x_1, x_2 \in M$, there exists an element $k \in K$ taking the fiber P_{x_1} over x_1 to the fiber P_{x_2} over x_2.

Thus, K acts transitively on the set of fibers, whereas G acts transitively inside the single fibers.

The fiber-transitive K-action induces a transitive K-action on M, hence M is a homogeneous space diffeomorphic to some left coset space K/H.

From now: Assume that K acts fiber-transitively on a principal G-bundle P, and write the base space M as K/H, where $H \subset K$.

For short, we call $P \to M$ a homogeneous principal G-bundle

The intertwining homomorphism $\lambda : H \to G$

$$\text{left action of } H \text{ on } P \longleftrightarrow \text{right action of } G \text{ on } P$$

Together, they induce a Lie group homomorphism $\lambda\colon H \to G$.

Fix a point $p_0 \in P$, $x_0 := \pi(p_0)$ its projection under $\pi\colon P \to M$. Recall that the isotropy subgroup of x_0 is defined by

$$H = \{h \in K \mid hx_0 = x_0\}.$$

By definition of the induced action of K on M, the condition $hx_0 = x_0$ can be reformulated as

$$\pi(hp) = \pi(p), \quad p \in P_{x_0}.$$

$\Rightarrow$ the isotropy H of x_0 is the group of elements $h \in K$ that map the fiber P_{x_0} over x_0 to itself,

$\Rightarrow$ for $h \in H$, the element $L_h(p_0) = hp_0$ stays in the same fiber as p_0,

$\Rightarrow$ there also exists a unique element $a \in G$ that maps p_0 to $L_h(p_0)$, since G acts freely and transitively in the fibers of P,

$\Rightarrow$ we obtain a map $\lambda\colon H \to G$ by setting $\lambda(h) = a$, where $a \in G$ is the unique element satisfying $L_h(p_0) = R_a(p_0)$.

Theorem and Definition.

The map

$$\lambda\colon H \longrightarrow G, \quad h \longmapsto \lambda(h),$$

where $\lambda(h) \in G$ is uniquely characterised by

$$L_h(p_0) = R_{\lambda(h)}(p_0), \quad h \in H, \tag{1}$$

defines a Lie group homomorphism $\lambda\colon H \to G$. We call it the intertwining homomorphism.

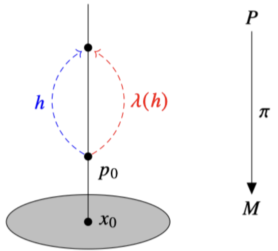


Illustration of the intertwining homomorphism $\lambda\colon H \to G$.

The intertwining homomorphism $\lambda : H \to G$

Important for geometry:

Let $P \to K/H$ be a homogeneous principal G-bundle with intertwining homomorphism $\lambda : H \to G$. Then P is K-equivariantly isomorphic to the associated bundle

$$K \times_\lambda G := (K \times G)/\sim, \quad (k,g) \sim (kh, \lambda(h)^{-1}g).$$

Moreover, the following conditions are equivalent:

1. The symmetry group K acts transitively on the total space P.
2. The intertwining homomorphism $\lambda : H \to G$ is surjective.
3. There exists a closed normal subgroup $L \triangleleft H$ such that P is K-equivariantly diffeomorphic to K/L.

Under this identification, the structure group is $G \cong H/L$ and the bundle projection is the canonical map $K/L \to K/H$.

This was exactly the situation encountered for 3-(α, δ)-Sasaki manifolds!

Important for connections (later):

Theorem.

The Lie algebra homomorphism $\lambda_* : \mathfrak{h} \to \mathfrak{g}$ corresponding to the intertwining homomorphism $\lambda \colon H \to G$ is uniquely characterised by

$$\widehat{X}_{p_0} = \widetilde{\lambda_*(X)}_{p_0}, \quad X \in \mathfrak{h},$$

where $\widehat{X}$ is the fundamental vector field on P corresponding to $X \in \mathfrak{h}$ and $\widetilde{\lambda_*(X)}$ is the fundamental vector field on P corresponding to $\lambda_*(X) \in \mathfrak{g}$.

Considering left and right fundamental VFs simultaneously cannot be avoided!

What is a connection?

Definition (Vertical Subspace).

Let $\pi: P \to M$ be a G-principal bundle. The distribution VP defined by

$$V_pP = \ker \mathrm{d}_p\pi = \{X \in T_pP \mid \mathrm{d}_p\pi(X) = 0\} \subset T_pP, \quad p \in P$$

is called the *vertical distribution* on the G-principal bundle P.

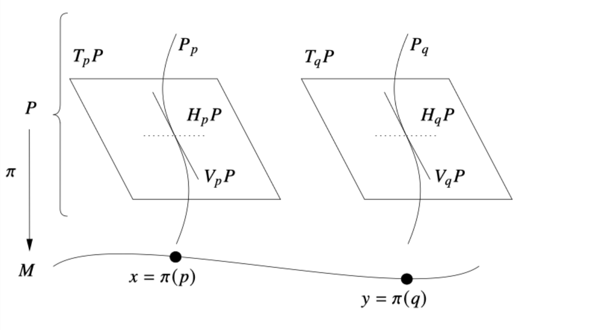
Fact: $\exists$ natural vector space isomorphism $\mathfrak{g} \to V_pP$ given by $X \longmapsto \tilde{X}_p$, the (right) fundamental VF induced by $X \in \mathfrak{g}$.

There is no canonical choice of a complement!

Definition (Ehresmann connection).

A *connection* Γ on a G-principal bundle P is a distribution HP on P, called *horizontal distribution*, satisfying the following conditions:

- $T_pP = H_pP \oplus V_pP$ for all $p \in P$,
- $\mathrm{d}_pR_a(H_pP) = H_{pa}P$ for all $p \in P,\ a \in G$.



Visualization of the Ehresmann connection on a principal fiber bundle $P \to M$.

Goal: characterise a connection Γ on a principal G-bundle P given by a horizontal distribution HP in terms of a $\mathfrak{g}$-valued one-form $\omega\colon TP \to \mathfrak{g}$

Let $X \in T_pP$ be a tangent vector at $p \in P$ with unique decomposition

$$X = X^H + X^V, \quad X^H \in H_pP,\ X^V \in V_pP.$$

We define the *connection one-form* $\omega\colon TP \to \mathfrak{g}$ associated with the connection Γ as follows:

$$\omega_p(X) = \omega_p(X^H + X^V) = \omega_p(X^H + \tilde{A}_p) = A \in \mathfrak{g}, \quad X \in T_pP.$$

Lemma.

The connection one-form ω associated with Γ satisfies

- $\omega(\tilde{A}) = A, \quad A \in \mathfrak{g},$
- $R_a^*\omega = \mathrm{Ad}_G(a^{-1}) \circ \omega, \quad a \in G,$

where $R_a\colon P \to P$ denotes the right action of $a \in G$ on P.
Conversely, $\forall\, \omega\colon TP \to \mathfrak{g}$ satisfying these two identities, $\exists_1$ connection HP on P s.t. the connection one-form associated with it agrees with ω,

$$H_pP = \ker \omega_p = \{X \in T_pP \mid \omega_p(X) = 0\}, \quad p \in P.$$

Principal G-bundle $(P, \pi, M; G)$	Choice of Rep. (V, ϱ) $\longrightarrow$	**Associated Vector Bundle** $E := P \times_G V$
Equivariant, horizontal k-forms on P $\Lambda^k_{\mathsf{EH}}(P, V)$	Isomorphism ι $\Lambda^k_{\mathsf{EH}}(P, V) \cong \Lambda^k(M, E)$	k-forms on M with values in E $\Lambda^k(M, E)$
Connection 1-form $A\colon TP \to \mathfrak{g}$ (Note: A is not horizontal!)	induces	Connection (cov. derivative) $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ $\nabla \circ \iota = \iota \circ D_A\big\vert_{\Lambda^0_{\mathsf{EH}}(P,V)}$
Exterior covariant differential $D_A\colon \Lambda^k_{\mathsf{EH}}(P, V) \to \Lambda^{k+1}_{\mathsf{EH}}(P, V)$	equivalent $(d_\nabla \circ \iota = \iota \circ D_A)$	Exterior covariant differential $d_\nabla\colon \Lambda^k(M, E) \to \Lambda^{k+1}(M, E)$
Curvature 2-form $F^A := D_A A \in \Lambda^2_{\mathsf{EH}}(P, \mathfrak{g})$	equivalent	Curvature operator $\mathcal{R}^\nabla(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$
$P \to M$ is flat, $P = M \times G$	equivalent	∇ on $P \times_G V$ is flat, $\mathcal{R}^\nabla = 0$

Example: The Ambrose-Singer connection

Let K/H be a *reductive* homogeneous space with reductive decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$, i.e. $\mathrm{Ad}_H(\mathfrak{m}) \subset \mathfrak{m}$. The distribution

$$H_k K = \mathrm{d}_e L_k(\mathfrak{m}), \quad k \in K,$$

where $L_k \colon K \to K$ denotes left multiplication in K, defines a horizontal distribution on the H-principal bundle $\pi \colon K \to K/H$:

- $V_k K = \ker \mathrm{d}_k \pi = \mathrm{d}_e L_k(\mathfrak{h})$ follows easily from $\ker \mathrm{d}_e \pi = \mathfrak{h}$,
- $T_k K = \mathrm{d}_e L_k(\mathfrak{k})$ implies that $T_k K = \mathrm{d}_e L_k(\mathfrak{h}) \oplus \mathrm{d}_e L_k(\mathfrak{m}) = V_k K \oplus H_k K$,
- Ad_H-invariance of $\mathfrak{m}$ implies that, for $h \in H$, we have

$$\begin{aligned} \mathrm{d}_k R_h(H_k K) &= \mathrm{d}_k R_h(\mathrm{d}_e L_k(\mathfrak{m})) = \mathrm{d}_e(L_k \circ R_h)(\mathfrak{m}) \\ &= \mathrm{d}_e L_{kh}(\mathrm{Ad}_H(h^{-1})(\mathfrak{m})) \subset \mathrm{d}_e L_{kh}(\mathfrak{m}) = H_{kh} K. \end{aligned}$$

For dimensional reasons, this implies that $\mathrm{d}_k R_h(H_k K) = H_{kh} K$. Hence, the distribution HK does indeed define a connection on $\pi \colon K \to K/H$, called the *Ambrose-Singer connection*. It is K-invariant by construction.

Suppose now that K acts on the principal G-bundle P fiber-transitively.

Definition (Invariance of a Connection).

A connection HP on P is called K-*invariant* if

$$\mathrm{d}_p L_k(H_p P) = H_{kp} P$$

for all $p \in P$ and all $k \in K$.

Recall: A horizontal distribution HP induces a unique connection one-form $\omega \colon TP \to \mathfrak{g}$ via the relation

$$H_p P = \ker \omega_p, \quad p \in P.$$

Proposition (Invariance via Connection One-Form).

A connection HP on P is K-invariant iff the corresponding connection one-form ω satisfies $L_k^* \omega = \omega$ for all $k \in K$.

Sketch of Proof

Theorem (Wang, 1958).

Let K be a connected Lie group acting fiber-transitively on a G-principal bundle P over a homogeneous space K/H and let H be the isotropy subgroup of $x_0 = \pi(p_0)$.
Then, there is a one-to-one correspondence between the set of K-invariant connections ω on P and the set of linear maps $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$ satisfying

1. $\Lambda(X) = \lambda_*(X), \quad X \in \mathfrak{h}$,
2. $\Lambda \circ \mathrm{Ad}_H(h) = \mathrm{Ad}_G(\lambda(h)) \circ \Lambda, \quad h \in H$,

where $\lambda\colon H \to G$ is the Lie group homomorphism defined by

$$hp_0 = p_0\lambda(h), \quad h \in H.$$

The correspondence is given by

$$\Lambda(X) = \omega_{p_0}(\widehat{X}_{p_0}), \quad X \in \mathfrak{k},$$

where $\widehat{X} \in \mathfrak{X}(P)$ is the fundamental vector field on P corresponding to $X \in \mathfrak{k}$.

We know how to obtain a linear map $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$ from a K-invariant connection form ω on P.

A Crucial Observation

The following result is key to proving Wang's theorem:

Lemma.

The assignment

$$T\colon \mathfrak{k} \times \mathfrak{g} \to T_{p_0}P, \quad (X, A) \mapsto \widehat{X}_{p_0} + \tilde{A}_{p_0},$$

where $\widehat{X}$ and $\tilde{A}$ denote the fundamental vector fields corresponding to $X \in \mathfrak{k}$ and $A \in \mathfrak{g}$, respectively, yields a surjective linear map with kernel

$$\ker T = \{(X, -\lambda_*(X)) \in \mathfrak{k} \times \mathfrak{g} \mid X \in \mathfrak{h}\}.$$

- proving linearity of T is straightforward,
- the rank-nullity theorem implies $\dim \operatorname{im} T = \dim K + \dim G - \dim \ker T$. We further have $\dim P = \dim K - \dim H + \dim G$. Altogether

$$\dim \operatorname{im} T = \dim P + \dim H - \dim \ker T.$$

- the statement then follows from the fact that $\lambda_* : \mathfrak{h} \to \mathfrak{g}$ is uniquely characterised by $\widehat{X}_{p_0} = \left(\widetilde{\lambda_*(X)}\right)_{p_0}$ for $X \in \mathfrak{h}$

Putting It All Together

Based on our observations, we

- associate to every K-invariant connection form $\omega\colon TP \to \mathfrak{g}$ the linear map

$$\Lambda\colon \mathfrak{k} \to \mathfrak{g}, \quad X \mapsto \omega_{p_0}(\widehat{X}_{p_0}) \tag{2}$$

- associate to every linear map $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$ satisfying $\Lambda(X) = \lambda_*(X),\ X \in \mathfrak{h}$ and $\Lambda \circ \mathrm{Ad}_H(h) = \mathrm{Ad}_G(\lambda(h)) \circ \Lambda,\ h \in H$ the one-form

$$\omega_p(\xi) = \mathrm{Ad}_G(a)(\Lambda(X) + A), \tag{3}$$

  where $(R_a \circ L_k)(p) = p_0$ and $\mathrm{d}_p(R_a \circ L_k)(\xi) = \widehat{X}_{p_0} + \tilde{A}_{p_0}$.

It is easy to see that the one-form $\omega\colon TP \to \mathfrak{g}$ defined by (3) and obtained from Λ satisfies $\omega_{p_0}(\widehat{X}_{p_0}) = \Lambda(X)$, i.e. the correspondence established above is indeed one-to-one.

What About the Conditions on $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$?

Q: Where do the identities $\Lambda(X) = \lambda_*(X)$ and $\Lambda \circ \mathrm{Ad}_H(h) = \mathrm{Ad}_G(\lambda(h)) \circ \Lambda$ come in to play?

First, the choices of $k \in K$ and $a \in G$ satisfying $(R_a \circ L_k)(p) = p_0$ are *not* unique.

If $\tilde{k} \in K$ and $\tilde{a} \in G$ are two further elements satisfying $(R_{\tilde{a}} \circ L_{\tilde{k}})(p) = p_0$, one checks that $h = \tilde{k}k^{-1}$ is in the isotropy subgroup H of x_0 and satisfies

$$\tilde{k} = hk \quad \text{and} \quad \tilde{a} = a\lambda(h)^{-1}.$$

A few computations demonstrate that

$$\mathrm{d}_p(R_{\tilde{a}} \circ L_{\tilde{k}})(\xi) = \widehat{Y}_{p_0} + B^\sharp_{p_0}, \quad Y = \mathrm{Ad}_H(h)(X),\ B = \mathrm{Ad}_G(\lambda(h))(A)$$

$$\begin{aligned}
\mathrm{Ad}_G(\tilde{a})(\Lambda(Y) + B) &= \mathrm{Ad}_G(\tilde{a})((\Lambda \circ \mathrm{Ad}_H(h))(X) + \mathrm{Ad}_G(\lambda(h))(A)) \\
&= \mathrm{Ad}_G(\tilde{a})((\mathrm{Ad}_G(\lambda(h)) \circ \Lambda)(X) + \mathrm{Ad}_G(\lambda(h))(A)) \\
&= \mathrm{Ad}_G(\tilde{a}\lambda(h))(\Lambda(X) + A) = \mathrm{Ad}_G(a)(\Lambda(X) + A)
\end{aligned}$$

Q: Where do the identities $\Lambda(X) = \lambda_*(X)$ and $\Lambda \circ \mathrm{Ad}_H(h) = \mathrm{Ad}_G(\lambda(h)) \circ \Lambda$

Some Consequences

The Reductive Case

Theorem (Wang, 1958).

Let K act fiber-transitively on a principal G-bundle P over a *reductive* homogeneous space K/H and let H be the isotropy subgroup of x_0. Then, there is a one-to-one correspondence between the set of K-invariant connections ω on P and the set of linear maps $\Lambda\colon \mathfrak{m} \to \mathfrak{g}$ satisfying

$$\Lambda \circ \mathrm{Ad}_H(h)|_{\mathfrak{m}} = \mathrm{Ad}_G(\lambda(h)) \circ \Lambda, \quad h \in H.$$

The correspondence is given by

$$\Lambda(X) = \omega_{p_0}(\widehat{X}_{p_0}), \quad X \in \mathfrak{m},$$

where $\widehat{X} \in \mathfrak{X}(P)$ is the fundamental vector field on P corresponding to $X \in \mathfrak{m}$.

- **Main idea:** Associate to every linear map $\Lambda\colon \mathfrak{m} \to \mathfrak{g}$ the linear map $\tilde{\Lambda}\colon \mathfrak{k} \to \mathfrak{g}$ defined by $\tilde{\Lambda}(X) = \lambda_*(X_{\mathfrak{h}}) + \Lambda(X_{\mathfrak{m}})$ and observe that

$$\lambda_* \circ \mathrm{Ad}_H(h) = \mathrm{Ad}_G(\lambda(h)) \circ \lambda_*, \quad h \in H$$

- Ambrose-Singer connection corresponds to $\Lambda = 0$!

Proposition (Inv. Cov. Derivatives on Red. Hom. Spaces).

Let K be Lie group and P be a K-invariant G-structure on the *reductive* homogeneous space K/H.
Then, there is a one-to-one correspondence between the set of K-invariant covariant derivatives ∇ on the tangent bundle of K/H (that are determined by connections on P) and the set of linear maps $\Lambda\colon \mathfrak{m} \to \mathfrak{g}$ satisfying

$$(\Lambda \circ \mathrm{Ad}_H(h))(X) = \lambda(h)\Lambda(X)\lambda(h)^{-1}, \quad h \in H,\ X \in \mathfrak{m}.$$

The correspondence is given by

$$\Lambda(X) = \mathbf{v}_o^{-1} \circ (\nabla_{\widehat{X}} - \mathcal{L}_{\widehat{X}})_o \circ \mathbf{v}_o, \quad X \in \mathfrak{m},$$

where $\widehat{X} \in \mathfrak{X}(K/H)$ is the fundamental vector field corresponding to $X \in \mathfrak{m}$ and $\mathbf{v}_o\colon \mathbb{R}^n \to T_o(K/H), \alpha \mapsto \alpha^i v_i$ is the vector space isomorphism induced by the linear frame $(o, (v_1, \ldots, v_n))$ at the origin $o \in K/H$.

- **Main idea:** Connection ω on frame bundle and induced covariant derivative ∇ on tangent bundle are related via

$$(\nabla_X - \mathcal{L}_X)_x = \mathbf{v} \circ \omega_{\mathbf{v}}(\tilde{X}_{\mathbf{v}}) \circ \mathbf{v}^{-1}, \quad x \in K/H,\ \mathbf{v} \in \mathcal{L}_x(K/H).$$

Why *Ambrose-Singer* Connection?

Proposition (Torsion and Curvature).

- $T_{\mathfrak{m}}(X,Y) = \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X,Y]_{\mathfrak{m}}$;
- $R_{\mathfrak{m}}(X,Y) = [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)]_{\mathfrak{gl}(\mathfrak{m})} - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \mathrm{ad}_{\mathfrak{h}}([X,Y]_{\mathfrak{h}})$,

In particular, the Ambrose-Singer connection satisfies $T_{\mathfrak{m}}(X,Y) = -[X,Y]_{\mathfrak{m}}$ and $R_{\mathfrak{m}}(X,Y)Z = -[[X,Y]_{\mathfrak{h}}, Z]_{\mathfrak{k}}$.

But, even more importantly, one shows that

- the torsion and curvature tensor fields of a K-invariant connection are K-invariant tensor fields
- and K-invariant tensor fields are parallel with respect to the AS connection!
- The AS connection is exactly the one appearing in the Ambrose-Singer Homogeneity Theorem!

Applications

What do we use this for?

- IA, Jonas Henkel, *The Laplace-Beltrami spectrum on Naturally Reductive Homogeneous Spaces*
- IA, Leandro Cagliero, Jonas Henkel, *Explicit Laplace Spectra of Homogeneous Principal Bundles*
- IA, Jordan Hofmann, Marie-Amélie Lawn, *Invariant spinors on homogeneous spheres*
- IA, Giulia Dileo, Giulia; Leander Stecker, *Homogeneous non-degenerate* 3-(α, δ)*-Sasaki manifolds and submersions over quaternionic Kähler spaces*
- Dream: Generalise these notions to biquotients