Codifferential Calculi on Quantum Homogeneous Spaces

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Conventions

- $\Bbbk$ - field of $\mathrm{char}(\Bbbk) \neq 2$ (e.g. $\Bbbk = \mathbb{C}$).
- Δ_M - right coaction of a right comodule M.
- ${}_M\Delta$ - left coaction of a left comodule M.
- q - non-zero complex number which is not a root of unity.

Motivation

In noncommutative geometry, smooth structures are captured by first-order differential calculi.

Definition. A *first-order differential calculus (FODC)* over a $\Bbbk$-algebra A consists of an A-bimodule Ω^1 and a derivation $d\colon A \to \Omega^1$ such that the map

$$A \otimes A \to \Omega^1, \quad a \otimes b \mapsto ad(b)$$

is surjective.

A case that is of particular interest is when A is a *quantum group* (i.e. a certain type of Hopf algebra). In practice, a quantum group A often comes with a dual quantum group U, in the sense that one has a Hopf algebra pairing

$$\langle -,- \rangle \colon A \times U \to \Bbbk$$

(e.g. $A = \mathscr{O}_q(G)$ and $U = U_q(\mathfrak{g})$ for a complex simple Lie algebra $\mathfrak{g}$). In this case one can ask whether a FODC over A induces a corresponding dual structure over U. This dual structure is a first-order codifferential calculus.

First-order codifferential calculi

Let C be a coalgebra (over $\Bbbk$). If M is a bicomodule over C, then a *coderivation* is a linear map $\delta\colon M \to C$ such that

$$\Delta(\delta(m)) = (\delta \otimes \mathrm{id})(\Delta_M(m)) + (\mathrm{id} \otimes \delta)({}_M\Delta(m))$$

for all $m \in M$.

Definition. A *first-order codifferential calculus (FOCC)* over C consists of a C-bicomodule $\mathscr{W}_1$ and a coderivation $\delta\colon \mathscr{W}_1 \to C$ such that the map

$$\mathscr{W}_1 \to C \otimes C, \quad w \mapsto (\mathrm{id} \otimes \delta)({}_M\Delta(w))$$

is injective.

The universal FOCC

Over any coalgebra C, the universal FOCC is defined. The underlying C-bicomodule is $\mathscr{W}_1^u := C \otimes C/\mathrm{im}(\Delta)$ and the coderivation is

$$\delta^u\colon \mathscr{W}_1^u \to C, \quad [c \otimes d] \mapsto \varepsilon(c)d - \varepsilon(d)c$$

where ε is the counit of C.

Theorem (Doi '81). For any C-bicomodule $\mathscr{W}_1$ and coderivation $\delta\colon \mathscr{W}_1 \to C$, there exists a unique C-bicolinear map $\varphi\colon \mathscr{W}_1 \to \mathscr{W}_1^u$ such that the diagram

$$\begin{tikzcd} \mathscr{W}_1 \arrow[r, dashed, "\varphi"] \arrow[dr, "\delta"'] & \mathscr{W}_1^u \arrow[d, "\delta^u"] \\ & C \end{tikzcd}$$

commutes.

In addition, it holds that $(\mathscr{W}_1, \delta)$ is a FOCC if and only if φ is injective. Thus, FOCC's over C correspond (up to isomorphism) to C-sub-bicomodules of $\mathscr{W}_1^u$.

The maximal prolongation

A FODC can universally be extended to a dg algebra. This is the noncommutative geometry-analog of passing from 1-forms to higher forms on a manifold. Dually, a FOCC can universally be extended to a *dg coalgebra*.

Definition. A *dg coalgebra* is a coalgebra $D_\bullet = \bigoplus_{n=0}^{\infty} D_n$ with a differential $\delta_\bullet\colon D_\bullet \to D_{\bullet-1}$ (i.e. $(D_\bullet, \delta_\bullet)$ is a chain complex) such that the coproduct $\Delta\colon D_\bullet \to D_\bullet \otimes D_\bullet$ is a morphism of chain complexes, where $D_\bullet \otimes D_\bullet$ is endowed with the usual differential acting via

$$c \otimes d \mapsto \delta_\bullet(c) \otimes d + (-1)^{|c|} c \otimes \delta_\bullet(d)$$

for homogeneous elements $c, d \in D_\bullet$.

If $(\mathscr{W}_1, \delta)$ is a FOCC over C, then we call a dg coalgebra $(D_\bullet, \delta_\bullet)$ a *prolongation* of $(\mathscr{W}_1, \delta)$ if $D_0 = C$, $D_1 = \mathscr{W}_1$ and $\delta_\bullet|_{D_1} = \delta$.

Theorem (B' 26). For any FOCC $(\mathscr{W}_1, \delta)$, there exists a prolongation $(\mathscr{W}_\bullet^{\max}, \delta_\bullet^{\max})$ such that for any other prolongation $(D_\bullet, \delta_\bullet)$ there exists a unique morphism of dg coalgebras $\psi_\bullet\colon D_\bullet \to \mathscr{W}_\bullet^{\max}$ extending the identity on $\mathscr{W}_1 \oplus C$.

$$\begin{tikzcd} \cdots \arrow[r] & \mathscr{W}_3^{\max} \arrow[r] & \mathscr{W}_2^{\max} \arrow[dr] & & \\ & & & \mathscr{W}_1 \arrow[r, "\delta"] & C \\ \cdots \arrow[r] & D_3 \arrow[uu, dashed, "\psi_3"] \arrow[r] & D_2 \arrow[uu, dashed, "\psi_2"] \arrow[ur] & & \end{tikzcd}$$

Equivariant codifferential calculi

Let U be a Hopf algebra, let $L \subseteq U$ be a right coideal subalgebra (i.e. $\Delta(L) \subseteq L \otimes U$) and write $C := U \otimes_L \Bbbk$. Write ${}^C_U\mathrm{Mod}^C$ for the category of C-bicomodules and left U-modules M such that

$${}_M\Delta(X \triangleright m) = \Delta(X)\,{}_M\Delta(m), \quad \Delta_M(X \triangleright m) = \Delta(X)\Delta_M(m)$$

for all $m \in M$ and $X \in U$.

Definition. A FOCC $(\mathscr{W}_1, \delta)$ over C is called *U-equivariant* if $\mathscr{W}_1$ is an object in ${}^C_U\mathrm{Mod}^C$ and δ is U-linear.

Definition. A *quantum tangent space (qTS)* is a subspace $T^1 \subseteq C^+ = \ker(\varepsilon)$ such that $LT^1 \subseteq T^1$ and

$$\Delta(T^1) \subseteq (T^1 \oplus \Bbbk[1]) \otimes C.$$

Theorem (B '26). Assume that U is faithfully flat as a left- and right L-module, and let $T^1 \subseteq C^+$ be a qTS. Then $\mathscr{W}_1 := U \otimes_L T^1$ together with

$$\delta\colon U \otimes_L T^1 \to C, \quad X \otimes_L t \mapsto X \triangleright t$$

is a U-equivariant FOCC over C. Moreover, every U-equivariant FOCC over C is isomorphic to a FOCC as above for a unique qTS T^1.

In the following, we write $t^+ := t - \varepsilon(t)[1]$ for an element $t \in C$. In addition, consider the map

$$\check{\delta}\colon T^1 \otimes T^1 \to U \otimes_L C, \quad [X] \otimes t \mapsto X_{(1)} \otimes_L S(X_{(2)}) \triangleright t$$

where $T^1 \subseteq C^+$ is a qTS.

Theorem (B '26). Assume that L is a sub bialgebra of U. Let T^1 be a qTS with *trivial right C-coaction*, i.e.

$$((-)^+ \otimes \mathrm{id})(\Delta(t)) = t \otimes [1]$$

for all $t \in T^1$. Then the maximal prolongation of $\mathscr{W}_1 := U \otimes_L T^1$ takes the form

$$U \otimes_L \mathfrak{C}(T^1, \check{R})$$

where $\check{R} := \check{\delta}^{-1}(U \otimes_L T^1)$, and $\mathfrak{C}(T, \check{R})$ is a *quadratic coalgebra*.

Codifferential calculi on quantized projective spaces

Let $U := U_q(\mathfrak{sl}_{n+1})$ the quantized universal enveloping algebra of $\mathfrak{sl}_{n+1}$ with generators $E_1, \ldots, E_n, F_1, \ldots, F_n$ and $K_1^{\pm 1}, \ldots, K_n^{\pm 1}$. Write $L := U_q(\mathfrak{l}_S)$ for the subalgebra generated by $E_2, \ldots, E_n, F_2, \ldots, F_n$ and $K_1^{\pm 1}, \ldots, K_n^{\pm 1}$, and $C := U_q^c(\mathbb{C}P^n) := U \otimes_L \mathbb{C}$. We consider a qTS as follows:

$$T^{(0,1)} = \mathrm{span}_{\mathbb{C}}\{[E_i E_{i-1} \ldots E_1] \mid 1 \leq i \leq n\}.$$

Write $\mathscr{W}^q_{(0,1)}(\mathbb{C}P^n)$ for the corresponding FOCC. Then its maximal prolongation can be written as

$$\mathscr{W}^q_{(0,\bullet)}(\mathbb{C}P^n) = U \otimes_L \Lambda_q^c(T^{(0,1)})$$

here $\Lambda_q^c(T^{(0,1)})$ denotes the dual coalgebra of the algebra with generators $e^1, \ldots, e^n$ and relations

$$e^i e^j = -q^{-1} e^j e^i, \quad 1 \leq i \leq j \leq n.$$

Duality between differential- and codifferential calculi

For $C = U \otimes_L \Bbbk$ as above, and L is a sub bialgebra. Consider the finitary dual $B := C^\circ$. In many examples B is a *quantum homogeneous space* (see [1, Section 4.3] for precise statements for when this happens). For instance

$$U_q^c(\mathbb{C}P^n)^\circ \cong \mathscr{O}_q(\mathbb{C}P^n).$$

In particular, B is a left coideal subalgebra of a *total space Hopf algebra A* (which is contained in U°). We write $H := A/B^+A$, where $B^+ = \ker(\varepsilon|_B)$. Let $\mathscr{W}_1$ be a U-equivariant FOCC over C with qTS T^1 which has trivial right C-coaction and consider $V^1 := B^+/I^1$, where

$$I^1 = \{b \in B^+ \mid \forall t \in T^1 : b(t) = 0\}.$$

Then $\Omega^1 := A \square_H V^1$ is a left covariant FODC over B, and we have the following result, which holds in the context of any irreducible quantum flag manifold (e.g. $\mathscr{O}_q(\mathbb{C}P^n), \mathscr{O}_q(\mathrm{Gr}(m,n))$).

Theorem (B '26). Under assumptions made precise in [1, Theorem 6.9], the maximal prolongation of Ω^1 is given as

$$\Omega^\bullet_{\max} = A \square_H \mathfrak{A}(V^1, \check{R}^\perp).$$

Here $\mathfrak{A}(V^1, \check{R}^\perp)$ is the quadratic dual algebra of $\mathfrak{C}(T^1, \check{R})$:

$$\mathfrak{A}(V^1, \check{R}^\perp) := T(V^1)/\langle \check{R}^\perp \rangle,$$

where $\check{R}^\perp$ is the orthogonal complement of $\check{R}$ under the pairing

$$V^1 \otimes V^1 \times T^1 \otimes T^1 \to \Bbbk, \quad \langle v \otimes w, s \otimes t \rangle = v(s)w(t).$$

References

[1] J. Benner. Codifferential calculi on Quantum Homogeneous Spaces. Preprint, arXiv:2606.09687

[2] Y. Doi. Homological coalgebra. J. Math. Soc. Japan 33 (1981), no. 1, 31–50.

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