Noncommutative integral forms and differential smoothness revisited

Tomasz Brzeziński

Swansea University and University of Białystok

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Initial thoughts or motivation

- Understanding smoothness of noncommutative algebras based on classical Poincaré duality provided by the Hodge star on a compact oriented (Riemannian) manifold $\mathcal{M}$.
- The duality is provided by the relation between differential forms (de Rham complex) and **integral forms** studied for supermanifolds by Manin.
- Complex of integral forms is the complex corresponding to a flat **right** connection on $\mathcal{M}$.
- Any finite-dimensional compact oriented (super)manifold admits a flat right connection with certain uniqueness property.

is closed with respect to the supercommutator. The Frobenius theorem for supermanifolds asserts the same thing as the theorem for purely even manifolds: integrability of a distribution is equivalent to the property that locally $\mathcal{T}$ be freely generated by vector fields of the form $(\partial/\partial x^a, \partial/\partial \xi^b)$, where (x^a, ξ^b) can be extended to a local system of coordinates. This is true for the holomorphic and the infinitely differentiable cases.

The proof can be carried out by induction on $p+q$, where $\operatorname{rk}\mathcal{T} = p|q$. The cases $1|0$ and $0|1$ form the content of Theorem 3.

5. Connections on a locally free sheaf. Let $\mathcal{E}$ be a locally free sheaf on a supermanifold M. A connection on $\mathcal{E}$ is defined by a covariant differential $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1 M$ which satisfies the Leibniz formula

$$\boxed{\nabla(ae) = (-1)^{(\tilde{a}+1)\tilde{e}} e \otimes da + (-1)^{\tilde{a}} a \nabla e\,.}$$

The signs are adapted to the relations: $\Omega^1 = \Omega^1_{\mathrm{odd}}$ and $\tilde{\nabla} = 1$. The concepts of the de Rham sequence of the connection, the curvature, and integrability are all defined as in the purely even case. A specifically odd construction is the definition of a right connection; the next subsection is devoted to this.

§ 5. Right Connections and Integral Forms

1. Right connections. We continue to work in the category of differentiable or analytic supermanifolds and set $K = \mathbb{R}$ or $\mathbb{C}$, respectively. We will consider a connection on a locally free sheaf $\mathcal{E}$ on M to be a covariant derivative, i.e., a K-bilinear mapping $\Delta\colon \mathcal{T}M \otimes_K \mathcal{E} \to \mathcal{E}$. This can be extended to a K-bilinear mapping $\Delta_l\colon \mathcal{D}_1 \otimes_K \mathcal{E} \to \mathcal{E}$, where $\mathcal{D}_1$ is the sheaf of differentiable operators on M of order ≤ 1, by setting $\Delta_l(f \otimes e) = fe$, where f is a local function viewed as an operator of order zero, and by extending the operator by additivity. The axioms of a connection, defined using Δ_l, take on a simpler form; in addition to the K-bilinearity of Δ_l, we have for any local operator X of order ≤ 1, any function f and any section e:

(a) $\Delta_l(f \otimes e) = fe$,

(b) $\boxed{\Delta_l(X \otimes fe) = \Delta_l(X \circ f \otimes e)};$

(c) $\Delta_l(fX \otimes e) = f\Delta_l(X \otimes e)$.

The second identity is the Leibniz formula. $X \circ f$ denotes the product of X and f as operators; the circle is written to distinguish the result from the application of X fo f: $X \circ f = (-1)^{\tilde{f}\tilde{X}} fX + Xf$. We introduce the idea of a right connection in an analogous way:

2. Definition. A *right connection* on $\mathcal{E}$ is defined to be a K-bilinear mapping $\Delta_r\colon \mathcal{E} \otimes \mathcal{D}_1 \to \mathcal{E}$ with the following properties:

(a) $\Delta_r(e \otimes f) = ef$;

(b) $\boxed{\Delta_r(e \otimes X \circ f) = \Delta_r(e \otimes X)f};$

(c) $\Delta_r(e \otimes fX) = \Delta_r(ef \otimes X)$.

In works on physics one writes $e\overleftarrow{X}$ instead of $\Delta_r(e \otimes X)$. If X is a vector field, the operator $\nabla_r(X)$ of right covariant differentiation along X is defined by the formula $\nabla_r(X)e = (-1)^{\tilde{X}\tilde{e}}\Delta_r(ef \otimes X)$.

On $\mathcal{O}_M$ there is an obvious left connection: $\Delta_l(X \otimes f) = Xf$. In turns out that on $\operatorname{Ber} M = (\operatorname{Ber}\Omega^1_{\mathrm{odd}} M)^*$ there is a canonical right connection.

3. Proposition. *On* $\operatorname{Ber} M$ *there is a unique right connection* Δ_r *satisfying the following condition: for any local system of coordinates* $(x^a) = x$ *we have*

$$\boxed{\Delta_r\left(D^*(dx) \otimes \frac{\partial}{\partial x^a}\right) = 0 \qquad \textit{for all } a,}$$

where $D^*(dx)$ *is the local section of* $\operatorname{Ber} M$ *dual to* $D(dx^1, \dots, dx^{m+n})$.

Proof. The uniqueness of Δ_r follows from the fact that $D^*(dx)$ is a right $\mathcal{O}_M$-basis for $\operatorname{Ber} M$, while the $\partial/\partial x^a$ form a right $\mathcal{O}_M$-basis of $\mathcal{T}M$ (locally). Thus the axioms for a right connection uniquely define the action on all of $\mathcal{D}_1$. Therefore, in the domain of (x^a) we obtain the following formula for Δ_r:

$$\begin{aligned}
&\Delta_r\left(D^*(dx)\, f \otimes \left(\sum_a g^a \frac{\partial}{\partial x^a} + h\right)\right) \\
&\quad = \Delta_r\left(D^*(dx) \otimes \sum_a (-1)^{(\tilde{f}+\tilde{g}^a)\tilde{x}^a}\left[\frac{\partial}{\partial x^a} \circ (fg^a) - \frac{\partial (fg^a)}{\partial x^a}\right] + fh\right) \\
&\quad = -D^*(dx) \otimes \left[\sum_a (-1)^{(\tilde{f}+\tilde{g}^a)\tilde{x}^a} \frac{\partial (fg^a)}{\partial x^a} + fh\right],
\end{aligned}$$

which, it is not difficult to see, satisfies the axioms for a right connection.

The final point to check is that $\Delta_r\left(D^*(dy) \otimes (\partial/\partial y^b)\right) = 0$, where $y^b = y^b(x)$ is another coordinate system. In fact

$$dy^c = \sum dx^a \otimes \frac{\partial y^c}{\partial x^a}, \qquad \frac{\partial}{\partial y^b} = \sum \frac{\partial x^a}{\partial y^b}\frac{\partial}{\partial x^a},$$

The origins

- Semi-free dgas $(\Omega A, d)$ over A ($\Omega^{n+1}A = \Omega^n A \otimes_A \Omega^1 A$) correspond bijectively to A-corings C with a group-like element (Roiter '80)
- Flat connections on A w.r.t. a semi-free dga generated by C correspond to (right) comodules over C (B '03).
- (C, Δ, ε) is an A-coring iff $- \otimes_A C$ is a **comonad** on $\mathbf{M}_A$.
- $\mathbf{M}^C$ is (isomorphic to) the Eilenberg-Moore cat of $- \otimes_A C$.
- $- \otimes_A C$ is the left adjoint of $\mathrm{Hom}_A(C, -)$, hence
- $\mathrm{Hom}_A(C, -)$ is a **monad** on $\mathbf{M}_A$.
- The Eilenberg-Moore category of $\mathrm{Hom}_A(C, -)$ is the category of **contramodules** of C (Eilenberg-Moore, '65).
- W.r.t. semi-free dga, Manin's flat right connections correspond to contramodules of C (B '08).

Hom-connections (B '08)

Let A be an algebra, $(\Omega A, d)$ be a dga over A, $M \in \mathbf{M}_A$.

- A **hom-connection** on M is a $\mathbb{K}$-linear map

$$\nabla : \mathrm{Hom}_A(\Omega^1 A, M) \to M,$$

such that, for all $f \in \mathrm{Hom}_A(\Omega^1 A, M)$ and $a \in A$,

$$\nabla(fa) = \nabla(f)a + f(da),$$

where $(fa)(\omega) := f(a\omega)$.

- Extension to $\nabla_n : \mathrm{Hom}_A(\Omega^{n+1} A, M) \to \mathrm{Hom}_A(\Omega^n A, M)$, by

$$\nabla_n(f)(\omega) := \nabla(f\omega) + (-1)^{n+1} f(d\omega),$$

where $(f\omega)(\omega') := f(\omega \wedge \omega')$.

Flatness. Integral forms.

- The **curvature** of ∇, $F := \nabla \circ \nabla_1$ is a right A-module map.
- ∇ is **flat** provided $F = 0$.
- $\nabla_n \circ \nabla_{n+1} \simeq \mathrm{Hom}_A(\Omega^n A, F)$ [B '08].
- If ∇ is flat there is a chain complex $(\mathrm{Hom}_A(\Omega^n A, M), \nabla_{n-1})$.
- In particular, for $M = A$, $\mathcal{I}_n A := \mathrm{Hom}_A(\Omega^n A, A)$ are A-bimodules and given a flat hom-connection, $(\mathcal{I}_n A, \nabla_{n-1})$ is called a complex of **integral forms**.
- The canonical map $\Lambda : A \to \mathrm{coker}(\nabla)$ is called the **∇-integral**.

Uniqueness and existence of integral calculus

Of course:

- Integral calculus is not unique (true of a diff calculus too).
- A might not admit a complex of integral forms with respect to a given dga.

Theorem (B-El Kaoutit-Lomp '10)

A right A-module M admits a hom-connection with respect to the universal dga if and only if M is an injective module.

So, in the case of the universal dga, $(\mathcal{I}A, \nabla)$ may exist only if A is a self-injective algebra.

Why integral (toy example, B '11)?

- A is a superalgebra of functions on the supercircle $S^{1|1}$:

$$a(x,\theta) = a_0(x) + a_1(x)\theta,$$

where $a_i : [0,1] \to \mathbb{R}$, $a_i(0) = a_i(1)$, $\theta^2 = 0$.

- $\Omega^1 A$: generated by dx, $d\theta$, $da = \partial_x(a)dx + \partial_\theta(a)d\theta$,

$$\partial_x a(x,\theta) := a_0'(x) + a_1'(x)\theta, \quad \partial_\theta a(x,\theta) := a_1(x).$$

- The flat hom-connection (divergence):

$$\nabla(f)(x,\theta) = f_{x,0}'(x) - f_{\theta,1}(x) + f_{\theta,0}'(x)dx\theta,$$

$$f(dx)(x,\theta) = f_{x,0}(x) + f_{x,1}(x)\theta, \;\; f(d\theta)(x,\theta) = f_{\theta,0}(x) + f_{\theta,1}(x)\theta.$$

- The ∇-integral Λ is the Berezin integral:

$$\Lambda(a) = \int_0^1 a_1(x)dx,$$

Why integral?

Theorem (B-El Kaoutit-Lomp '10)

Let $\Omega^1 A$ be a left covariant diff. calculus on a Hopf algebra A.

(1) Let $\{\omega_1, \ldots, \omega_n\}$ be a left invariant basis for $\Omega^1 A$. Let $\{\xi_1, \ldots, \xi_n\}$ be its right dual basis and let $\{\lambda_1, \ldots, \lambda_n\}$ be its left dual basis. Then

$$\begin{aligned} \nabla : \mathrm{Hom}_A(\Omega^1 A, A) &\longrightarrow A \\ f &\longmapsto \sum_i \left(S^2 \circ \lambda_i \circ d \circ S^{-2}\right)(f(\omega_i)) \end{aligned}$$

is a unique hom-connection on A such that $\nabla(\xi_i) = 0$.

(2) Assume that $\Omega^1 A$ extends to a dga such that ∇ is flat, and that there is a right integral $\lambda : A \to \mathbb{K}$ on A. Then there exists a unique map $\varphi : \mathrm{coker}(\nabla) \to \mathbb{K}$ such that

$$\lambda = \varphi \circ \Lambda,$$

where $\Lambda : A \to \mathrm{coker}(\nabla)$ is the ∇-integral on A.

4. Integral forms. Integral forms are defined to be sections of the sheaf

$$\Sigma M = \mathrm{Ber}\, M \otimes_{\mathcal{O}_M} S(\mathcal{T}M\Pi) = \mathcal{H}om_{\mathcal{O}_M}(\Omega^{\bullet}_{\mathrm{odd}}M, \mathrm{Ber}\, M)\,.$$

Here $\mathcal{T}M\Pi$ is an $\mathcal{O}_M$-module with the same left multiplication by $\mathcal{O}_M$ as $\mathcal{T}M$ but with reversed parity. By analogy with the exterior differential on ΩM, one can define a differential $\delta\colon \Sigma M \to \Sigma M$ with $\delta^2 = 0$. To begin with, we introduce the definition in local coordinates. In the domain of the coordinates (x^a) we have $\Sigma M = \mathrm{Ber}\, M \otimes_K K\left[\frac{\partial}{\partial x^a}\Pi\right]$; we set

$$\delta^x = \sum_a \nabla_r\left(\frac{\partial}{\partial x^a}\right) \otimes_K \frac{\partial}{\partial(\frac{\partial}{\partial x^a}\Pi)}(-1)^{\widetilde{x}^a}\,.$$

5. Proposition. *The definition $\delta = \delta^x$ does not depend upon the choice of the coordinates (x^a); also $\delta^2 = 0$.*

Proof. To check that $\delta^2 = 0$ we apply the criterion of § 4 of Chapter 5. The operators $\nabla_r\left(\frac{\partial}{\partial x^a}\right)$ and $\partial/\partial\left(\frac{\partial}{\partial x^a}\Pi\right)$ have opposite parity. It remains to show that the first ones also commute pairwise. Let $X = \partial/\partial x^a$ and $Y = \partial/\partial x^b$. Then for $M = m|n$

$$\begin{aligned}\nabla_r(X)(D^*(dx)\,f) &= (-1)^{\widetilde{X}(m+\widetilde{f})}\Delta_r(D^*(dx)\,f \otimes X)\\ &= (-1)^{\widetilde{X}m+1}D^*(dx)\,Xf\,,\end{aligned}$$

whence $[\nabla_r(X), \nabla_r(Y)] = (-1)^{(\widetilde{X}+\widetilde{Y})m}\nabla_r([X,Y]) = 0$.

To check that the definition is correct, we first show that on $\mathrm{Ber}\, M \otimes \mathcal{T}M\Pi$ the operator δ^x does not depend on (x^a); more precisely,

$$\delta^x(\omega \otimes X\Pi) = (-1)^{\widetilde{\omega}+\widetilde{X}}\Delta_r(\omega \otimes X)\,.$$

In fact, let $\omega = D^*(dx)\,f$ and $X = \sum g^a\,(\partial/\partial x^a)$. Then

$$\begin{aligned}\delta^x(\omega \otimes X\Pi) &= \left(\sum_b \nabla_r\left(\frac{\partial}{\partial x^b}\right) \otimes_K \frac{\partial}{\partial(\frac{\partial}{\partial x^b}\Pi)}(-1)^{\widetilde{x}^b}\right)\left(\sum_a D^*(dx)\,fg^a \otimes_K \frac{\partial}{\partial x^a}\Pi\right)\\ &= \sum_a (-1)^{\widetilde{x}^a+(\widetilde{x}^a+1)(\widetilde{\omega}+\widetilde{g}^a)}\nabla_r\left(\frac{\partial}{\partial x^a}\right)(D^*(dx)\,fg^a)\\ &= \sum_a (-1)^{\widetilde{x}^a+(\widetilde{x}^a+1)(\widetilde{\omega}+\widetilde{g}^a)}(-1)^{\widetilde{x}^a(\widetilde{\omega}+\widetilde{g}^a)}\Delta_r\left(D^*(dx)\,fg^a \otimes \frac{\partial}{\partial x^a}\right)\end{aligned}$$

Finally, we prove that for $F, G \in K\left[\frac{\partial}{\partial x^b}\Pi\right]$ we have the Leibniz formula:

$$\delta^x(\omega \otimes_K FG) = \delta^x(\omega \otimes_K F)G + (-1)^{GF}\delta^x(\omega \otimes_K G)F\,.$$

From this it will follow by induction on the degree of F, beginning with one, that $\delta^x(\omega \otimes F)$ does not depend on (x^a).

In fact:

$$\begin{aligned}\delta^x(\omega \otimes FG) &= \left[\sum_a \nabla_r\left(\frac{\partial}{\partial x^a}\right) \otimes_K \frac{\partial}{\partial\left(\frac{\partial}{\partial x^a}\Pi\right)}(-1)^{\widetilde{x}^a}\right](\omega \otimes FG)\\ &= \sum_a \nabla_r\left(\frac{\partial}{\partial x^a}\right)\omega \otimes (-1)^{\widetilde{\omega}(\widetilde{x}^a+1)+\widetilde{x}^a}\left[\frac{\partial F}{\partial\left(\frac{\partial}{\partial x^a}\Pi\right)}G + (-1)^{\widetilde{F}(\widetilde{x}^a+1)}F\frac{\partial G}{\partial\left(\frac{\partial}{\partial x^a}\Pi\right)}\right];\end{aligned}$$

the decomposition into two summands inside the brackets corresponds to the right side of the Leibniz formula. □

6. Remarks. (a) As usual, in the purely even case ($n = 0$) there is an isomorphism $\mathrm{Ber}\, M \to \Omega^m M$ which carries $D^*(dx)$ to $dx^1 \wedge \cdots \wedge dx^m$. (The parity of this isomorphism is $(-1)^m$ if we assume that $\Omega^m M$ has rank $1|0$.) The contraction operation of a vector field with a differentiable form permits us to construct an isomorphism $\Omega^m M \otimes \wedge^i(\mathcal{T}M) \xrightarrow{\sim} \wedge^{m-i}(\mathcal{T}^*M) = \Omega^{m-i}M$. Over the sheaf of purely even rings $\mathcal{O}_M$, one can identify $\wedge^i(\mathcal{T}M)$ with $S^i(\mathcal{T}M\Pi)$. Thus in the purely even case, integral forms essentially are the same as differential forms; in this case the differentials are also identified up to some signs.

(b) If the $\mathbb{Z}$-grading of the module of differential forms is chosen in the usual way, i.e., set $\mathcal{O}_M = \Omega^0$ and take $dx^a \in \Omega^1$, then it is natural to say that $S^i(\mathcal{T}M\Pi)$ has degree $-i$ and that δ increases degree by 1. The degree of $D^*(dx)$ can be decided in several possible ways. Since for even constants t we have $D^*(d(tx)) = t^{m-n}D^*(dx)$, one can consider $\mathrm{Ber}\, M = \Sigma_{m-n}M$. If one wishes to have the homology of the complex of sections of integral forms vanish in the same interval as the differential forms, then it is natural to consider $\mathrm{Ber}\, M = \Sigma_m M$:

$$\begin{gathered}\mathcal{O}_M \to \Omega^1 M \to \cdots \to \Omega^m M \to \Omega^{m+1}M \to \cdots\\ \cdots \to \Sigma_{-1}M \to \Sigma_0 M \to \Sigma_1 M \to \cdots \to \mathrm{Ber}\, M\,.\end{gathered}$$

7. Spencer sequence of a sheaf with right connection. Now let $\mathcal{E}$ be a locally free sheaf with right connection Δ_r on M. We set $\Sigma(\mathcal{E}) = \mathcal{E} \otimes_{\mathcal{O}_M} S(\mathcal{T}M\Pi)$. The preceding construction can be carried out for $\mathcal{E}$ instead of $\mathrm{Ber}\, M$. More precisely, in the domain of the coordinates (x^a) we set

Volume forms

Let A be an algebra.

- By an **n-dimensional differential calculus over A** we mean a differential graded algebra $(\Omega A, d)$ such that
  (a) $\Omega A = \oplus_{k=0}^{n} \Omega^k A$, with $\Omega^0 A = A$ and $\Omega^n A \neq 0$;
  (b) as an algebra, ΩA is generated by A and $d(A)$;
  (c) $\ker d \,|_A = \mathbb{K}.1$.
- ΩA **admits a volume form**, say ω, if $\Omega^n A$ is freely generated by ω as a right A-module.
- A volume form ω induces an algebra automorphism $\nu : A \to A$ and the right A-module isomorphism $\pi : \Omega^n A \to A$, given by

$$a\omega = \omega\nu(a), \qquad \pi(\omega a) = a, \qquad \text{for all } a \in A.$$

Integrable calculi

- $\mathcal{I}A := \oplus_{k=0}^{n} \mathcal{I}_k A$ is a right ΩA-module with multiplication

  $$(\varphi{\cdot}\omega')(\omega'') = \varphi(\omega' \wedge \omega''), \quad \omega' \in \Omega^k A,\ \omega'' \in \Omega^m A,\ \varphi \in \mathcal{I}_{k+m} A.$$

- A volume form $\omega \in \Omega^n A$ is said to be an **integrating form** if the maps,

  $$\Theta_k : \Omega^k A \to \mathcal{I}_{n-k} A, \qquad \omega' \mapsto \pi_\omega \wedge \omega', \qquad k = 1, 2, \ldots, n{-}1,$$

  are bijective. These are in fact A-bimodule isomorphisms, provided the left A-module structure of the $\mathcal{I}_k A$ is twisted by ν.
- A calculus admitting such a form is said to be **integrable**.

Integrable calculi and duality

Thus for an integrable calculus, the following diagram with isomorphisms of bimodules as vertical arrows is commutative

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, "\Theta_0"'] & \Omega^1 A \arrow[r, "d"] \arrow[d, "\Theta_1"'] & \Omega^2 A \arrow[r, "d"] \arrow[d, "\Theta_2"'] & \dots \arrow[r, "d"] & \Omega^{n-1} A \arrow[r, "d"] \arrow[d, "\Theta_{n-1}"'] & \Omega^n A \arrow[d, "\Theta_n"] \\
{}^{\nu}\mathcal{I}_n A \arrow[r, "\nabla_{n-1}"] & {}^{\nu}\mathcal{I}_{n-1} A \arrow[r, "\nabla_{n-2}"] & {}^{\nu}\mathcal{I}_{n-2} A \arrow[r, "\nabla_{n-3}"] & \dots \arrow[r, "\nabla_1"] & {}^{\nu}\mathcal{I}_1 A \arrow[r, "\nabla"] & {}^{\nu}A\,.
\end{tikzcd}$$

The bottom row is an integral complex.

The usual suspects

- The classical calculus on $\mathbb{C}[x_1, \ldots, x_n]$ is integrable.
- The 3D calculus on $\mathcal{O}_q(SU(2))$ is integrable.
- The 2D restriction of the 3D calculus to the standard quantum Podleś sphere is integrable.

The Gelfand-Kirillov dimension

- Let A be a finitely generated or affine algebra with generating subspace $\mathcal{V}$. Let us write $\mathcal{V}(n)$ for the subspace of A spanned by 1 and all words in generators of A of length at most n. The algebra A is said to have **polynomial growth** if there exist $c \in \mathbb{R}$ and $p \in \mathbb{N}$ such that $\dim \mathcal{V}(n) \leq cn^p$ for all sufficiently large n.
- The **Gelfand-Kirillov dimension** of A is a real number defined as

$$\mathrm{GKdim}(A) := \inf\{p \in \mathbb{R} \mid \dim \mathcal{V}(n) \leq n^p,\ n \gg 0\}.$$

Differentially smooth algebras

Definition (B-Sitarz '17)

An affine algebra with integer Gelfand-Kirillov dimension n is said to be **differentially smooth** if it admits an n-dimensional integrable differential calculus.

Expected examples

- The polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ is differentially smooth.
- Coordinate algebras of the quantum group $SU_q(2)$, the standard quantum Podleś sphere and the quantum Manin plane are smooth.
- The algebra $\mathbb{C}[x, y]/\langle xy \rangle$ is not differentially smooth.

Main advantage and main obstacle

Main advantage
Differential smoothness is a *constructive* notion: one needs to construct a specific calculus.

Main obstacle
Differential smoothness is a *constructive* notion: very few general procedures can be expected; case-by-case studies required.

Tensoriality

Theorem (TB & Lomp '16)

Let $(\Omega A, d)$ be an m-dimensional integrable calculus on A and $(\Omega B, d)$ be an n-dimensional integrable calculus on B. If all $\Omega^k A$ and $\Omega^k B$ are finitely generated projective as right A, resp. B-modules, then the tensor product calculus $(\Omega(A \otimes B), d)$

$$\Omega(A \otimes B)^k = \bigoplus_i \Omega^i A \otimes \Omega^{k-i} B,$$

is an integrable calculus on $A \otimes B$.

Corollary

Under the projectivity assumptions on ΩA and ΩB, if A and B are differentially smooth and one of them has GK-dimension not exceeding 2, then $A \otimes B$ is differentially smooth.

Skew polynomial rings

Theorem (B-Lomp '16)

Let A be a differentially smooth algebra with integrating calculus $(\Omega A, d)$ such that all $\Omega^k A$ are finitely projective right A-modules. Let $\sigma : \Omega A \to \Omega A$ be an automorphism of differential graded algebras. Then the skew polynomial rings $A[z;\sigma]$ and $A[z,z^{-1};\sigma]$ are differentially smooth.

Example (Karaçuha-Lomp '14)

The algebra $\mathbb{K}_{\mathbf{q}}[x_1,\ldots,x_n]$ generated by $x_1,\ldots,x_n$,

$$x_i x_j = q_i x_j x_i, \qquad \text{for all } i < j,$$

where $q_i \in \mathbb{K}^*$, is differentially smooth.

Integrability and principality

Theorem (B-Sitarz '17)

Let A be a principal H-comodule algebra with the coinvariant subalgebra B. Let ΩA be an H-covariant n-dimensional integrable calculus with integrating form ω. Assume that:

(a) $\Omega B = (\Omega A)^{coH}$;

(b) ω *is invariant, i.e.* $\rho^{\Omega^n A}(\omega) = \omega \otimes 1$, *and hence* $\omega \in \Omega^n B$;

(c) *for all* $k = 1, \ldots, n-1$, *if* $\omega' \in \Omega^k A$ *has the property that, for all* $\omega'' \in \Omega^{n-k} B$, $\omega' \wedge \omega'' \in \Omega^n B$, *then* $\omega' \in \Omega^k B$.

Then ΩB is an integrable differential calculus and ω is its integrating form.

The assumptions mean that both A and B have integrable calculi of equal dimensions. This limits the applicability to actions of finite quantum groups.

Quantum lens spaces

The coordinate algebra $\mathcal{O}(L_q(N;1,N))$ of the quantum lens space $L_q(N;1,N)$ is a $*$-algebra generated by ξ, ζ subject to the relations

$$\xi\zeta = q^l \zeta\xi, \qquad \xi\zeta^* = q^l \zeta^*\xi, \qquad \zeta\zeta^* = \zeta^*\zeta,$$

$$\xi\xi^* = \prod_{m=0}^{l-1}(1 - q^{2m}\zeta\zeta^*), \qquad \xi^*\xi = \prod_{m=1}^{l}(1 - q^{-2m}\zeta\zeta^*),$$

where $q \in (0,1)$ [Hong-Szymański '03].

Theorem (B-Sitarz '17)

The restriction $\Omega(L_q(N;1,N))$ of the 3D-calculus $\Omega(SU_q(2))$ to $\mathcal{O}(L_q(N;1,N))$ is a three-dimensional integrable differential calculus. Consequently, $\mathcal{O}(L_q(N;1,N))$ is a differentially smooth algebra.

Unexpected examples: the quantum cone

The quantum cone algebra, defined as a complex $*$-algebra generated by $a = a^*$ and b, b^*,

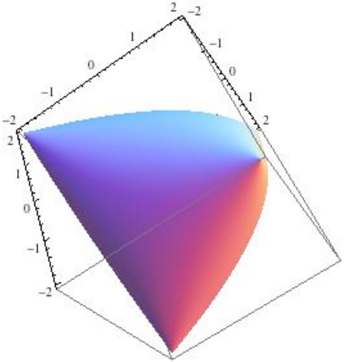
$$ab = q^N ba + \kappa[N]_q b, \quad bb^* = \prod_{l=0}^{N-1} (q^{-l} a + \kappa[-l]_q),$$

$$b^* b = \prod_{l=1}^{N} (q^l a + \kappa[l]_q),$$

is smooth provided $q \neq 1$.

Unexpected examples: the noncommutative pillow

The pillow orbifold:



non-smooth points (corners).

Unexpected examples: the noncommutative pillow

- Noncommutative torus $\mathcal{O}(\mathbb{T}^2_\theta)$: generated by unitary U, V with relation $UV = \lambda VU$, where $\lambda = \exp(2\pi\theta i)$, θ-irrational.
- The $\mathbb{Z}_2$-action:

$$U \mapsto U^*, \qquad V \mapsto V^*.$$

- The invariant subalgebra $\mathcal{O}(P_\theta)$, generated by $x = U + U^*$ and $y = V + V^*$ with relations:

$$yx - \bar{\lambda}xy = \mu z, \quad zx - \lambda xz = -\mu y, \quad yz - \lambda zy = -\mu x,$$

$$x^2 + y^2 + \bar{\lambda}z^2 - xzy = 2(1 + \bar{\lambda}^2),$$

where $\mu = 2i\sin(2\pi\theta)$ and $z = UV^* + U^*V$, is differentially smooth .

Hopf algebra examples

Theorem (B '15)

Every affine pointed Hopf domain (over an algebraically closed field of characteristic 0) of Gelfand-Kirillov dimension two that is not a polynomial identity ring is differentially smooth.

Following [Wang-Zhang-Zhuang '13] all such Hopf algebras are Ore extensions of the polynomial ring $\mathbb{K}[x]$:

$$A[q, r; p(x)] := \mathbb{K}\langle x, y\rangle / \langle qxy + ry + p(x) - yx\rangle,$$

or the Laurent polynomial ring $\mathbb{K}[x, x^{-1}]$:

$$A[q, \pm; p(x)] := \mathbb{K}\langle x, x^{-1}, y\rangle / \langle qx^{\pm 1}y + p(x) - yx\rangle,$$

$q, r, \in \mathbb{K}$, $q \neq 0$, $p(x) \in \mathbb{K}[x]$.

The smoothness of Ore extensions

Theorem

(1) The algebra $A[q, r; p(x)]$ is differentially smooth in the following three cases:

(a) $q = 1$, $r = 0$ with no restriction on $p(x)$;

(b) $q = 1$, $r \neq 0$ and $p(x) = c$, $c \in \mathbb{K}$;

(c) $q \neq 1$, $p(x) = c(x + \frac{r}{q-1})$, $c \in \mathbb{K}$ with no restriction on r.

(2) The algebras $A[1, +; p(x)]$, $A[q, +; c]$, $A[q, -; c(x - qx^{-1})]$ are differentially smooth.

The algebras in the Wang, Zhang & Zhuang classification are: $A[1, 0; x]$, $A[q, +; 0]$ and $A[1, +; x^n - x]$, hence they are all smooth by the above theorem.

Recent examples

- Andrés Rubiano in his PhD thesis (2024) and a series of papers ('24-26, some jointly with A. Reyes) analysed several examples of algebras proving their diff smoothness.
- Some classes of AS regular algebras (of GK dimension 5, for example), diffusion algebras, PBW extensions of algebras etc.
- He also derived no-go theorems, showing that some restrictions imposed on calculi will clash with the integrability.

Questions

- Is the non-standard Podleś sphere $S^2_{q,s}$ diff smooth?
- Do the covariant first-order diff calculi on $S^2_{q,s}$ classified in [Heckenberger-Kolb '02] extend to integrable calculi? They are inner and hence there is a hom-conn $f \mapsto f(\Xi)$, which is flat if $d\Xi + \Xi^2 = 0$ [B '08].
- With recent developments in the theory of quantum flag manifolds by Ó Buachalla et al., can one deduce diff smoothness for more general quantum homogenous spaces (not just for quotients by finite quantum subgroups as in the case for the lens spaces)?
- Are quantum teardrops [B-Frairfax '12] diff smooth?
- What can be said about subalgebras of differentially smooth algebra? How about Ore extensions?
- How about generalised Weyl algebras introduced by Bavula?