Quantum $SL(2,\mathbb{R})$ and dynamical quantum $SU(2)$

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Main result [DC - Dzokou Talla '26]

Let $0<q<1$.

Then $SL_q(2,\mathbb{R})$ and $\widehat{SU_q}^{dyn}(2)$ are unitarily monoidally equivalent :

$$\mathrm{Rep}^u(SL_q(2,\mathbb{R})) \overset{\otimes}{\cong} \mathrm{Rep}^u(\widehat{SU_q}^{dyn}(2))$$

Plan:

0. Setting
1. Quantum $SL(2,\mathbb{R})$
2. Towards dynamical quantum $SU(2)$
3. Dynamical quantum $SU(2)$
4. Categorical considerations

⓪ Setting : unitarity

*) Interested in unitary representations : Modules are Hilbert spaces

$\langle -, - \rangle$: linear $\}$ anti-linear; pos. def. + complete

→ can be ∞-dim! ⇒ analytic tools

→ auxiliary : pre-Hilbert spaces

→ $\langle -, - \rangle$ carried along when, e.g., inducing ⇒ tweaks & twists!

*) Interested in (possibly) non-unital ("non-compact") $*$-algebras ("real forms")

$$\begin{cases} a^{**} = a \\ (a+\lambda b)^* = a^* + \bar{\lambda} b^* \\ (ab)^* = b^* a^* \end{cases}$$

$$\langle a\xi, \eta \rangle = \langle \xi, a^* \eta \rangle.$$

*) Sometimes $*$-algebras have norm (C^*-algebras) or weak topology (vN algebras)

→ "Auxiliary" $*$-algebra's are useful :

$\boxed{G \text{ Lie group}, \ \mathfrak{g} = \mathrm{Lie}(G)}$

$$\begin{array}{ccc} A & \leftrightarrow & \mathcal{A} \\ \updownarrow & & \updownarrow \\ \mathsf{A} & \leftrightarrow & \mathfrak{A} \end{array}$$

, e.g.

$$\begin{array}{ccc} \mathcal{O}(G) & \leftrightarrow & C_c(G) \text{ (pointwise)} \\ \updownarrow & & \text{Non-unital } *\text{-algebra} \\ & & \updownarrow \\ C_0(G) & \leftrightarrow & L^\infty(G) \\ C^*\text{-algebra} & & \text{vN algebra} \end{array}$$

, or

$$\begin{array}{ccc} U(\mathfrak{g}) & \leftrightarrow & C_c(G) \text{ (convolution)} \\ \updownarrow & & \text{Non-unital } *\text{-algebra} \\ & & \updownarrow \\ C^*(G) & \leftrightarrow & L(G) \\ C^*\text{-algebra} & & \text{vN algebra} \end{array}$$

⓪ Setting: Categories

We want to understand representation categories, e.g. of the following kind:

→ **Non-unitary**: $\mathbb{G} = (\overset{\mathcal{O}(\mathbb{G})}{\overset{\shortparallel}{A}}, \Delta)$ Hopf algebra,
$B_1, B_2, \dots$ right A-comodule algebras

$$\Rightarrow \; {}_{B_1}\mathcal{M}od^{A}_{B_2} = \begin{array}{l} B_1\text{-}B_2\text{-bimods } \mathcal{V} \\ \quad + \\ \text{compatible } A\text{-comodule structure on } \mathcal{V} \end{array}$$

morphisms = equivariant linear maps

We have $\underset{B_2}{\otimes} : {}_{B_1}\mathcal{M}od^{A}_{B_2} \times {}_{B_2}\mathcal{M}od^{A}_{B_3} \to {}_{B_1}\mathcal{M}^{A}_{B_3}$

→ **Unitary**: $\mathbb{G} = (\overset{L^\infty(\mathbb{G})}{\overset{\shortparallel}{\mathcal{A}}}, \Delta)$ LCQG,
DC-DeRo '24 $\quad B_1, B_2, \dots$ $\mathbb{G}$-vN-algebras

$$\Rightarrow \; {}_{B_1}\mathrm{Hilb}^{\mathcal{A}}_{B_2} = \begin{array}{l} B_1\text{-}B_2\text{- correspondences } \mathcal{H} \\ \quad + \\ \text{compatible unitary } \mathbb{G}\text{-rep. on } \mathcal{H} \end{array}$$

morphisms = equivariant bounded linear maps ($\Rightarrow$ $*$-structure!)

We have $\underset{B_2}{\boxtimes} : {}_{B_1}\mathrm{Hilb}^{\mathcal{A}}_{B_2} \times {}_{B_2}\mathrm{Hilb}^{\mathcal{A}}_{B_3} \to {}_{B_1}\mathrm{Hilb}^{\mathcal{A}}_{B_3}$

① Quantum $SL(2,\mathbb{R})$: building blocks.

$A = \mathcal{O}_q(SU(2))$ = compact real form of $\mathcal{O}_q(SL(2))$

⇝ CQG Hopf $*$-algebra $(\Delta(a^*) = \Delta(a)^{*\otimes *})$ with

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(A) \ \underline{\text{unitary}}: \quad U^*U = I_2 = UU^*,$$

$$\text{so} \quad U^* = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = U^{-1} = \begin{pmatrix} S(\alpha) & S(\beta) \\ S(\gamma) & S(\delta) \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}$$

$B = \mathcal{O}_q(S^2) = \mathcal{O}_q(SO(2)\backslash SU(2)) \subseteq \mathcal{O}_q(SU(2))$ generated by matrix coefficients of

$$E = U^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U \quad \in M_2(A)$$

⇝ Right coideal $*$-subalgebra $B \subseteq A$ ⟹ right A-comodule $*$-algebra
(Equatorial Podleś sphere)

$C = \mathcal{O}_q(SO(2)) = A/AB_+$ with $B_+ = \mathrm{Ker}(\varepsilon_{|B}) \subseteq B$

⇝ Left quotient A-module $\dagger$-coalgebra $\begin{cases} c^{\dagger\dagger} = c \\ (c + \lambda d)^\dagger = c^\dagger + \bar{\lambda} d^\dagger \\ \Delta(c^\dagger) = \Delta^{opp}(c)^{\dagger\otimes\dagger} \\ (ac)^\dagger = a^\dagger c^\dagger \end{cases}$ via $\begin{matrix} [a]^\dagger = [a^\dagger] \\ a^\dagger = S(a)^* \end{matrix}$

[Koppinen '93]

① Quantum $SL(2,\mathbb{R})$: Unitary Doi-Koppinen modules

<u>Def</u> : A unitary right C-comodule V is a pre-Hilbert space with right C-comodule structure s.t.

$$\left(\langle w, v_{(0)} \rangle \, v_{(1)} \right)^{\dagger} = \langle v, w_{(0)} \rangle \, w_{(1)}$$

(and s.t. V is locally complete)

$\mathrm{Hilb}^C \simeq \mathrm{Rep}^u(SO_q(2)) \cong \mathrm{Hilb}^{\mathbb{Z}}$

($\mathbb{Z}$-graded Hilbert spaces)

(Müller - Schneider '99)

A unitary (B,C)-Doi-Koppinen module is a unitary right C-comodule with a $*$-rep. of B s.t.

$$\underbrace{(bv)_{(0)}}_{\in V} \otimes \underbrace{(bv)_{(1)}}_{\in C} = \underbrace{b_{(1)}}_{\in B} \underbrace{v_{(0)}}_{\in V} \otimes \underbrace{b_{(2)}}_{\in A} \underbrace{v_{(1)}}_{\in C} .$$

<u>Def</u> : We define the $*$-category

(DC - Dzokou Talla '24)

$$\mathrm{Rep}^u(SL_q(2,\mathbb{R})) := {}_B\mathrm{Hilb}^C = \{ V \mid V \text{ unitary } (B,C)\text{-DK-module} \}$$

with <u>bounded</u> DK-intertwiners as morphisms.

1 Quantum $SL(2,\mathbb{R})$: motivation

Let $I := U_q(so(2)) := \langle b = iq^{-1/2}(E - FK) \rangle \subseteq U_q(su(2)) =: U$

$\rightsquigarrow$ Left coideal $*$-subalgebra (Koornwinder '93)

We have a non-degenerate $*$-pairing

$$p : U_q(so(2)) \times \mathcal{O}_q(SO(2)) \longrightarrow \mathbb{C}$$

$$p : U_q(su(2)) \times \mathcal{O}_q(SU(2)) \longrightarrow \mathbb{C}$$

$p(X, [a]) = p(X, a)$

$\rightsquigarrow p(X, c^+) = \overline{p(X^*, c)}$

$\Rightarrow \quad \mathrm{Hilb}^C \subseteq {}_I\mathrm{Hilb}$ = pre-Hilbert spaces with $*$-rep. of I

$\Rightarrow \quad {}_B\mathrm{Hilb}^C \subseteq {}_{D(B,I)}\mathrm{Hilb}$ = pre-Hilbert spaces with $*$-rep. of $D(B,I)$

where $D(B,I) = \langle B, I \mid Xb = p(\underbrace{b_{(2)}}_{\in A}, \underbrace{X_{(1)}}_{\in U})\, \underbrace{b_{(1)}}_{\in B}\, \underbrace{X_{(2)}}_{\in I} \rangle$

But in a formal setting:

$$D(\underset{U_q(a \oplus n)}{\mathcal{O}_q(S^2)}, U_q(so(2))) \xrightarrow[q \to 1]{} U(sl(2,\mathbb{R}))$$

2) Towards dynamical quantum $SU(2)$: covariant bicomodules

Def: A unitary C-bicomodule V is a pre-Hilbert space with C-bicomodule structure s.t.

$$\left(\langle w, v_{(0)}\rangle v_{(1)}\right)^{+} = \langle v, w_{(0)}\rangle w_{(1)}$$

$$\left(v_{(-1)}\langle w, v_{(0)}\rangle\right)^{+} = w_{(-1)}\langle v, w_{(0)}\rangle$$

(+ locally complete)

A unitary A-covariant C-bi-comodule is a unitary C-bicomodule with a $*$-rep. of A s.t.

$$(av)_{(0)} \otimes (av)_{(1)} = a_{(1)} v_{(0)} \otimes a_{(2)} v_{(1)}.$$

$$\tau(a) = K^{-1/2} \triangleright a \triangleleft K^{1/2}$$

⚠ $$(av)_{(-1)} \otimes (av)_{(0)} = \tau(a_{(1)}) v_{(-1)} \otimes a_{(2)} v_{(2)}$$

Def: We define the $*$-category

$${}^{C}_{A}\mathrm{Hilb}^{C} = \{ V \mid V \text{ unitary } A\text{-covariant } C\text{-bi-comodule} \}$$

with bounded equivariant intertwiners as morphisms.

② Towards dynamical quantum $SU(2)$: unitary Takeuchi equivalence

Theorem:

We have an equivalence of $*$-categories

$$ {}^{C}_{A}\mathrm{Hilb}^{C} \longrightarrow {}_{B}\mathrm{Hilb}^{C} $$

$$ V \mapsto {}^{\circ}V = \{ v \in V \mid v_{(-1)} \otimes v_{(0)} = [1_A] \otimes v \} $$

with original C-comodule but

twisted B-representation

$$ b \triangleright v = \kappa(b)v \ , \quad \kappa(b) = b \triangleleft K^{-1/2}. $$

⚠ $\kappa : B \hookrightarrow A$

$\kappa(B) \neq B$

Note: ${}^{C}_{A}\mathrm{Hilb}^{C}$ has monoidal structure

$\Rightarrow \mathrm{Rep}^{u}(SL_q(2,\mathbb{R}))$ inherits it!

what is a direct construction?

② Towards dynamical quantum $SU(2)$: missing piece

Main result: Let $0 < q < 1$.

Then $SL_q(2,\mathbb{R})$ and $\widehat{SU}_q^{dyn}(2)$ are unitarily monoidally equivalent:

$$\begin{array}{ccc} \mathrm{Rep}^u(SL_q(2,\mathbb{R})) & \overset{\otimes}{\cong} & \mathrm{Rep}^u(\widehat{SU}_q^{dyn}(2)) \\ \int \,\| & & \|\!S \ \ ?? \\ {}_B\mathrm{Hilb}^C & \overset{\otimes}{\cong} & {}^C_A\mathrm{Hilb}^C \\ & \checkmark & \end{array}$$

③ Dynamical quantum $SU(2)$: definition of $*$-algebra

Consider 2 copies $\mathcal{F}_\lambda$, $\mathcal{F}_s$ of $*$-algebra $\mathcal{F} = \{\text{Functions } q^{\mathbb{Z}} \to \mathbb{C}\}$.

Def: We define $\mathcal{O}_q^{dyn}(SU(2))$ as the unital $*$-algebra generated by $\mathcal{F}_\lambda$, $\mathcal{F}_s$ and the entries of $U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

s.t. ① $U = \begin{pmatrix} U_{--} & U_{-+} \\ U_{+-} & U_{++} \end{pmatrix}$ is unitary

② $f(\lambda, s)\, U_{\epsilon,\nu} = U_{\epsilon,\nu}\, f(q^{-\epsilon}\lambda, q^{-\nu}s)$, $f \in \mathcal{F}_\lambda \hat{\otimes} \mathcal{F}_s$

③ $U^*_{\epsilon,\nu} = U_{-\epsilon,-\nu}\, \dfrac{\nu \cdot w_\nu^{1/2}(s)}{\epsilon\, w_\epsilon^{1/2}(\lambda)}$ $\qquad W_\pm(x) = \dfrac{q^{\pm 1}x + q^{\mp 1}x^{-1}}{x + x^{-1}}$

Remark: Originally $\mathcal{F}$ = meromorphic functions on $\mathbb{C}$, no $*$-structure. (Etingof – Varchenko '98)

③ Dynamical quantum $SU(2)$ as quantum groupoid

Put $\mathcal{F}_c = \{\text{Functions } f : q^{\mathbb{Z}} \to \mathbb{C} \text{ of Finite support}\} \subseteq \mathcal{F}$

$\Rightarrow$ (non-unital) $*$-subalgebra

$$\mathcal{A} = \mathcal{O}^{dyn}_{q,cpt}(SU(2)) := \mathcal{F}_{\lambda,c}\,\mathcal{F}_{\rho,c}\,\mathcal{O}^{dyn}_q(SU(2)) \subseteq \mathcal{O}^{dyn}_q(SU(2)).$$

Note: with $1\begin{pmatrix} x \\ y \end{pmatrix} = \delta_x(\lambda)\,\delta_y(\rho)$, we get

$$\mathcal{A} = \oplus\; {}^{x}_{y}\mathcal{A}^{w}_{z}\,, \quad {}^{x}_{y}\mathcal{A}^{w}_{z} = 1\begin{pmatrix} x \\ y \end{pmatrix} \mathcal{A}\, 1\begin{pmatrix} w \\ z \end{pmatrix},$$

so multiplication : ${}^{x}_{y}\mathcal{A}^{w}_{z} \times {}^{w}_{z}\mathcal{A}^{u}_{v} \to {}^{x}_{y}\mathcal{A}^{u}_{v}$

Theorem : There exist "partial" comultiplications

[DC-Timmermann '15]

[DC '16]

$${}_r\Delta_s : {}^{x}_{y}\mathcal{A}^{w}_{z} \to {}^{x}_{r}\mathcal{A}^{w}_{s} \otimes {}^{r}_{y}\mathcal{A}^{s}_{z}$$

making $\mathcal{A}$ into a partial compact quantum group

"$\Delta = \oplus_{r}\Delta_s$ $*$-hom and $\Delta(u_{\epsilon r}) = \sum_{k\in\pm} u_{\epsilon k}\otimes u_{k r}$, $\Delta\left(1\begin{pmatrix} x \\ y \end{pmatrix}\right) = \sum_{z} 1\begin{pmatrix} x \\ z \end{pmatrix}\otimes 1\begin{pmatrix} z \\ y \end{pmatrix}$."

③ Dynamical quantum $SU(2)$: Representations of $\widehat{SU_q^{dyn}(2)}$

<u>Def</u>: $\mathrm{Rep}^{``}(\widehat{SU_q^{dyn}(2)}) :=$ (non-degenerate) $*$-reps of $\mathcal{A}$ (so "$\sum_{x,y} 1\binom{x}{y} = id$")

Via $\Delta = \bigoplus_{r,s} {}_r\Delta_s$ and $\mathcal{H} \boxtimes \mathcal{G} := \bigoplus_r \mathcal{H} \otimes^r \mathcal{G}$ this is a monoidal $*$-category.

<u>Thm</u> [Stokman '03, DC-Dzokou Talla '18] We have $\mathrm{Rep}^{``}(\widehat{SU_q^{dyn}(2)}) \overset{\otimes}{\cong} {}^{C}_{A}\mathrm{Hilb}^{C}$

<u>Pf</u> (Sketch):

① ${}^{C}\mathrm{Hilb}^{C} \cong {}_{\mathcal{F}_c \otimes \mathcal{F}_c}\mathcal{H}\mathrm{ilb}$ since "$\mathrm{Spec}(ib) = \{[m]_q \mid m \in \mathbb{Z}\}$"

② Hence ${}^{C}_{A}\mathrm{Hilb}^{C} \cong$ (non-degenerate) $*$-reps of $\mathcal{F}_c \rtimes A \ltimes \mathcal{F}_c$

③ Construct $*$-iso $\mathcal{A} \cong \mathcal{F}_c \rtimes A \ltimes \mathcal{F}_c$

④ Dynamical quantum $SU(2)$: Main theorem

Main result: Let $0 < q < 1$.

Then $SL_q(2,\mathbb{R})$ and $\widehat{SU}_q^{dyn}(2)$ are unitarily monoidally equivalent:

$$\begin{array}{ccc}
\mathrm{Rep}^u(SL_q(2,\mathbb{R})) & \overset{\otimes}{\cong} & \mathrm{Rep}^u(\widehat{SU}_q^{dyn}(2)) \\
\checkmark \; \| & & \wr\| \; \checkmark \\
{}_B\mathrm{Hilb}^C & \overset{\otimes}{\cong} & {}^C_A\mathrm{Hilb}^C \\
 & \checkmark &
\end{array}$$

④ Categorical constructions.

The main theorem concerns finding good models of categories for computations (e.g. fusion rules).

In this last part: what are more intrinsic / conceptual ways in which these categories / models appear in the first place?

E.g.: When is a category a representation category?

And if operation: categories $\Rightarrow$ category,

then rep. category $\Rightarrow$ rep. category?

④ Categorical constructions: Examples

Ⓘ Tanaka - Krein

→ $(\mathcal{C}, \otimes)$ unitary $\otimes$-cat, $\mathcal{D} = \mathrm{Hilb}$ with unitary right $\mathcal{C}$-mod. structure
($\Leftrightarrow$ Fiber Functor)

$$\Rightarrow \mathcal{C} \overset{\otimes}{\cong} \mathrm{Rep}^u(\mathbb{G})$$ for compact quantum group $\mathbb{G} = (A, \Delta)$

⇝ Tannaka - Krein - Woronowicz !

→ More generally: $\mathcal{D}$ unitary right $\mathcal{C}$-mod. cat with $\mathcal{D} \cong \mathrm{Hilb}^S$, set S

$$\Rightarrow \mathcal{C} \overset{\otimes}{\cong} \mathrm{Rep}^u(\mathcal{G})$$ for partial compact quantum group $\mathcal{G}$

$\mathcal{G} = (A, \Delta, \{1\binom{s}{t}\}_{s,t \in S})$

(DC - Timmermann '15)

④ Categorical constructions: Examples (ct'd)

(II) Duality

$\mathbb{G} = (A, \Delta)$ CQG Hopf $*$-algebra, $\mathcal{C} = \mathrm{Rep}^u(\mathbb{G}) = \mathrm{Hilb}^A$, $\mathcal{D} = \mathrm{Hilb}$

then $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ and we can form $\mathcal{E} = \mathrm{End}_{\mathcal{C}}^{\mathrm{cont}}(\mathcal{D}) = \{(F, \alpha)\}$

Result : $(\mathcal{E}, \circ) \overset{\otimes}{\cong} ({}_A\mathrm{Hilb}, \otimes^{\mathrm{op}})$ (cf. Müger '03, EGNO '15, De Ro '25)

⇝ "Also true for PCQG $\mathbb{G} = (\mathcal{A}, \Delta)$ (over set $\mathbb{S}$) with $\mathcal{C} = \mathrm{Rep}_u(\mathbb{G})$, $\mathcal{D} = \mathrm{Hilb}^{\mathbb{S}}$" (unverified)

(III) Duality – variant

$\mathbb{G} = (A, \Delta)$ CQG Hopf $*$-algebra, $B \subseteq A$ coideal $*$-subalgebra, $C = A/AB_+$.

Then $\mathcal{C} = \mathrm{Rep}^u(\mathbb{G}) = \mathrm{Hilb}^A \curvearrowright \mathcal{D} = \mathrm{Hilb}^C \Rightarrow \mathcal{E} = \mathrm{End}_{\mathcal{C}}^{\mathrm{cont}}(\mathcal{D}) = \{(F, \alpha)\}$

Result : $(\mathcal{E}, \circ) \overset{\otimes}{\cong} ({}^C_A\mathrm{Hilb}^C, \square^{\mathrm{op}})$

④ Categorical constructions: Examples (ct'd)

IV Eilenberg - Watts.

$\mathbb{G} = (A, \Delta)$ CQG Hopf $*$-algebra, $B \subseteq A$ coideal $*$-subalgebra, $C = A/AB_+$.

Then $\mathcal{C} = \mathrm{Rep}^u(\mathbb{G}) = \mathrm{Hilb}^A \curvearrowright \mathcal{D} = \mathrm{Hilb}^C \Rightarrow \mathcal{E} = \mathrm{End}^{\mathrm{cont}}_{\mathcal{C}}(\mathcal{D}) = \{(F, \alpha)\}$

Takeuchi $\cong$ Hilb^A_B

$\hookrightarrow$ twisted compatibility for right B-module:

$$(\xi b)_{(0)} \otimes (\xi b)_{(1)} = \xi_{(0)} b_{(0)} \otimes \xi_{(1)} \tau(b_{(1)}).$$

Result: $(\mathcal{E}, \circ) \overset{\otimes}{\cong} \left({}_B\mathrm{Hilb}^A_B, \underset{B}{\boxtimes}^{op} \right)$ [De Ro '25]

④ Categorical constructions: Application

Theorem:

I $\begin{cases} ① \ \mathcal{O}_q(SU(2)) \text{ via TK on } \mathrm{Rep}^u(SU_q(2)) \curvearrowright \mathrm{Rep}^u(\{e\}) = \mathrm{Hilb} \\ ② \ \mathcal{O}^{dyn}_{q,cpt}(SU(2)) \text{ via TK on } \mathrm{Rep}^u(SU_q(2)) \curvearrowright \mathrm{Rep}^u(SO_q(2)) = \mathrm{Hilb}^C \end{cases}$

$\Longrightarrow \quad \mathrm{Rep}^u(SU_q(2)) \overset{\otimes}{\cong} \mathrm{Rep}^u(SU_q^{dyn}(2))$

cf. $SU_q^{dyn}(2)$ as "dynamical twist" of $SU_q(2)$ (Babelon '91, Etingof-Varchenko '98, Etingof-Nikshych '01, Stokman '03)

II ③ $\mathrm{End}^{cont,op}_{\mathrm{Rep}^u(SU_q^{dyn}(2))}(\mathrm{Rep}^u(SO_q(2))) \cong \mathrm{Rep}^u(\widehat{SU}_q^{dyn}(2))$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cong$ (vertical)

III ④ $\mathrm{End}^{cont,op}_{\mathrm{Rep}^u(SU_q(2))}(\mathrm{Rep}^u(SO_q(2))) \cong {}^{C}_{A}\mathrm{Hilb}^C \cong {}_B\mathrm{Hilb}^C = \mathrm{Rep}^u(SL_q(2,\mathbb{R}))$

IV ⑤ $\mathrm{End}^{cont,op}_{\mathrm{Rep}^u(SU_q(2))}(\mathrm{Rep}^u(SO_q(2))) \cong {}_B\mathrm{Hilb}_B^A \cong {}_B\mathrm{Hilb}^C = \mathrm{Rep}^u(SL_q(2,\mathbb{R}))$

Thank you !