On two notions of torsion and metric compatibility of connections in noncommutative geometry

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Abstract

We compare the notions of metric-compatibility and torsion of a connection in the frameworks of Beggs-Majid and Mesland-Rennie. It follows that for $*$-preserving connections, compatibility with a real metric in the sense of Beggs-Majid corresponds to Hermitian connections in the sense of Mesland-Rennie. If the calculus is quasi-tame, the torsion zero conditions are equivalent. A combination of these results proves the existence and uniqueness of Levi-Civita connections in the sense of Mesland-Rennie for unitary cocycle deformations of a large class of Riemannian manifolds, as well as the Heckenberger-Kolb calculi on all quantized irreducible flag manifolds.

Preliminaries on Beggs-Majid set up

Suppose A is an algebra and B is an A-comodule $*$-algebra. The symbol ${}^A_B\mathsf{Mod}_B$ will denote the category whose objects are B-bimodules and left A-comodules with the obvious compatibility relation and the morphisms are A-covariant maps that are B-bilinear.

Definition: A differential calculus on an algebra B is a differential graded algebra $(\Omega^\bullet(B) = \bigoplus_{k\geq 0}\Omega^k(B), \wedge, d)$ such that $\Omega^k(B)$ are objects of ${}^A_B\mathsf{Mod}_B$, $\Omega^0(B) = B$ and $\Omega^\bullet(B)$ is generated as an algebra by B and dB.

A differential calculus $(\Omega^\bullet, \wedge, d)$ on B is said to be a $*$-differential calculus if there exists a conjugate linear involution $* : \Omega^\bullet(B) \to \Omega^\bullet(B)$ which extends the map $* : B \to B$ such that

$$*(\Omega^k(B)) \subseteq \Omega^k(B),\ (d\omega)^* = d(\omega^*),\ (\omega\wedge\nu)^* = (-1)^{kl}\nu^*\wedge\omega^*$$

for all $\omega \in \Omega^k(B), \nu \in \Omega^l(B)$.

Definition: A metric is a pair $(g, (\ ,\ ))$ such that $g \in \Omega^1 \otimes_B \Omega^1$ is an A-coinvariant element and $(\ ,\ ) : \Omega^1 \otimes_B \Omega^1 \to B$ is a morphism of ${}^A_B\mathsf{Mod}_B$ such that for all $\omega \in \Omega^1$, the following equations hold:

$$((\omega,\ ) \otimes_B \mathrm{id})g = \omega = (\mathrm{id} \otimes_B (\ ,\omega))g.$$

A metric is said to be real if $g^\dagger = g$, where $\dagger = \mathrm{flip} \circ (* \otimes_B *)$.

Definition: A left connection on an object $\mathcal{E}$ of ${}^A_B\mathsf{Mod}_B$ is an A-covariant map $\nabla : \mathcal{E} \to \Omega^1 \otimes_B \mathcal{E}$ such that

$$\nabla(be) = b\nabla(e) + db \otimes_B e$$

for all $e \in \mathcal{E}$ and for all $b \in B$.

It is called a bimodule left connection if there exists a morphism $\sigma : \mathcal{E} \otimes_B \Omega^1 \to \Omega^1 \otimes_B \mathcal{E}$ in ${}^A_B\mathsf{Mod}_B$ such that

$$\nabla(eb) = \nabla(e)b + \sigma(e \otimes_B db)$$

for all $e \in \mathcal{E}$ and for all $b \in B$.

$*$-preserving connection

A bimodule connection on $\Omega^1(B)$ is said to be $*$-preserving if $\nabla \circ * = \sigma \circ \dagger \circ \nabla$, where $\dagger = \mathrm{flip} \circ (* \otimes_B *)$.

Levi-Civita Connection

A Levi-Civita connection on Ω^1 for a metric $(g, (\ ,\ ))$ is a left bimodule connection (∇, σ) such that it is compatible with g, i.e.,

$$\nabla_{\Omega^1\otimes_B\Omega^1}(g) := \Big(\nabla \otimes \mathrm{id} + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \nabla)\Big)(g) = 0$$

and is torsion-less, i.e., $\wedge \circ \nabla = d$.

Heckenberger–Kolb calculi

Let G be a compact connected simply connected simple Lie group with complexified Lie algebra $\mathfrak{g}$. Suppose that $\pi = \{\alpha_1, \cdots, \alpha_n\}$ is the set of simple roots of $\mathfrak{g}$ and $S = \pi \setminus \{\alpha_i\}$, where α_i appears with coefficient at most 1 in the highest root of $\mathfrak{g}$.

For $0 < q < 1$, the symbol $U_q(\mathfrak{g})$ will denote the Drinfeld-Jimbo quantized universal enveloping algebra while $\mathcal{O}_q(G)$ will denote the coordinate algebra of the dual compact quantum group G_q. Moreover, we have a quantum homogeneous space $\mathcal{O}_q(G/L_S)$, called the algebra of functions on the corresponding quantized irreducible flag manifold.

Theorem: Over any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, there exists a unique finite-dimensional left $\mathcal{O}_q(G)$-covariant differential $*$-calculus $\Omega_q^\bullet(G/L_S)$ of classical dimension.

For more details, we refer to [HK04, HK06] and references therein.

Unique connection on $\Omega_q^1(G/L_S)$

Theorem [DGKOB+21] Let $\Omega_q^\bullet(G/L_S)$ be the Heckenberger–Kolb Calculus on an irreducible flag manifold $\mathcal{O}_q(G/L_S)$. There exists a unique $\mathcal{O}_q(G)$-covariant connection ∇ on $\Omega_q^1(G/L_S)$.

Existence and Uniqueness of Levi-Civita connection on the Heckenberger-Kolb calculi

Theorem ([BGKOB24]) Let $(g, (\ ,\ ))$ be an $\mathcal{O}_q(G)$-covariant real metric on the space of one-forms of the Heckenberger–Kolb calculi. Then there exists a unique covariant bimodule connection (∇, σ) on $\Omega_q^1(G/L_S)$ which is torsionless and compatible with the metric $(g, (\ ,\ ))$.

The Levi-Civita Connection is $*$-preserving

Theorem: [BGG26] The Levi-Civita connection (∇, σ) on $\Omega_q^1(G/L_S)$ is $*$-preserving.

Unitary 2-cocycle deformation

Definition: If A is a Hopf algebra, then a convolution invertible map $\gamma : A \otimes A \to \mathbb{C}$ is called a 2-**cocycle** if

$$\gamma(g_{(1)} \otimes h_{(1)})\gamma\Big(g_{(2)}h_{(2)} \otimes k\Big) = \gamma\Big(h_{(1)} \otimes k_{(1)}\Big)\gamma\Big(g \otimes h_{(2)}k_{(2)}\Big)$$

for all $g, h, k \in A$ and $\gamma(h \otimes 1) = \epsilon(h) = \gamma(1 \otimes h)$ for all $h \in A$. If A is a Hopf $*$-algebra, then γ is said to be unitary if

$$\overline{\gamma\Big(a \otimes b\Big)} = \bar{\gamma}\Big(S(a)^* \otimes S(b)^*\Big).$$

If γ is a unitary 2-cocycle on A, then it is well-known that there exists a Hopf $*$-algebra $(A_\gamma, \cdot_\gamma, \Delta, \epsilon_\gamma, S_\gamma, *_\gamma)$ isomorphic to A as a coalgebra.
In the presence of a unitary cocycle γ, A-comodule $*$-algebras and objects of ${}^A_B\mathsf{Mod}_B$ can also be deformed. Concretely, if B is an A-comodule $*$-algebra, then we have an A_γ-comodule $*$-algebra B_γ with multiplication and involution defined as

$$a \cdot_\gamma b = \gamma\Big(a_{(-1)} \otimes b_{(-1)}\Big)a_{(0)}b_{(0)},\ b^{*_\gamma} := \bar{V}(b^*_{(-1)})\, b^*_{(0)}.$$

Here, $\overline{V} : A \to \mathbb{C}$ is the map defined as $\overline{V}(k) = \overline{\gamma}\Big(k_{(2)} \otimes S^{-1}(k_{(1)})\Big)$.
The cocycle deformation of an object $(M,\ {}^M\delta)$ of ${}^A_B\mathsf{Mod}_B$ is the pair $(\Gamma(M),\ {}^{M_\gamma}\delta)$ where $\Gamma(M) = M$ as a comodule equipped with B_γ-bimodule structures given by

$$b \cdot_\gamma m = \gamma\Big(b_{(-1)} \otimes m_{(-1)}\Big)b_{(0)} \cdot m_{(0)}, \quad m \cdot_\gamma b = \gamma\Big(m_{(-1)} \otimes b_{(-1)}\Big)m_{(0)} \cdot b_{(0)}.$$

Proposition: If $(\Omega^\bullet, \wedge, d)$ is an A-covariant differential $*$-calculus on B, then the unitary 2-cocycle deformation $(\Omega^\bullet_\gamma, \wedge_\gamma, d_\gamma)$ is an A_γ-covariant differential $*$-calculus on B_γ.
Proposition: If $(g, (\ ,\ ))$ is a real A-covariant metric on $\Omega^1(B)$ and γ a unitary 2-cocycle on A, then the cocycle-deformed metric $(g_\gamma, (\ ,\ )_\gamma)$ for the differential calculus $(\Omega^\bullet_\gamma, \wedge_\gamma, d_\gamma)$ is also real.

Levi-Civita connection for cocycle deformation

Theorem: [BM20] Assume that $(\nabla_{\Omega^1}, \sigma)$ is the unique A-covariant Levi-Civita connection for a covariant metric $(g, (\ ,\ ))$. Then $\nabla_{\Omega^1_\gamma} := \varphi^{-1}_{\Omega^1,\Omega^1} \circ \Gamma(\nabla_{\Omega^1})$ is a bimodule connection and is the unique A_γ-covariant Levi-Civita connection for the metric $(g_\gamma, (\ ,\ )_\gamma)$.

Here $\varphi^{-1}_{\Omega^1,\Omega^1}(v \otimes_B w) = \bar{\gamma}\Big(v_{(-1)} \otimes w_{(-1)}\Big)v_{(0)} \otimes_{B_\gamma} w_{(0)}$ for all $v, w \in \Omega^1$.

Theorem ([BGG26])

Let (∇, σ) be a $*$-preserving A-covariant bimodule connection on Ω^1 such that σ is an A-covariant map. Then the connection ∇_γ defined is also $*$-preserving.

Remark

A unitary 2-cocycle deformation of a classical Levi-Civita connection is $*$-preserving.

Mesland-Rennie ([MR25]) Set Up

Definition: A weak second order differential structure on B is a triplet (Ω^1, d, ψ) where (Ω^1, d) is a first order differential calculus on B and $\psi : \Omega^1 \otimes_B \Omega^1 \to \Omega^1 \otimes_B \Omega^1$ is a B-bilinear idempotent such that $JT^2_d \subseteq \mathrm{Ran}(\psi)$. Here JT^2_d is the space of second order junk forms.

Definition: A left connection ∇ on $\Omega^1(B)$ for a weak second order differential structure (Ω^1, d, ψ) is called torsionless in the sense of [MR25] if $(1-\psi)\nabla = d_\psi$ where

$$d_\psi : \Omega^1 \to \Omega^1 \otimes_B \Omega^1,\ d_\psi(\omega) = (1-\psi)(\widehat{\pi} \circ \delta \circ \pi^{-1})(\omega).$$

A differential calculi $(\Omega^\bullet, \wedge, d)$ is said to be quasi-tame if $\wedge : \Omega^1 \otimes_B \Omega^1 \to \Omega^2$ splits.

Comparison of torsion ([BGG26])

Let $(\Omega^\bullet, \wedge, d, s)$ be a quasi-tame differential calculus and ∇ is a connection on Ω^1, then ∇ is torsionless in the sense of [MR25] if and only if $\wedge \circ \nabla = d$.

Levi-Civita connection in [MR25] set up

Definition: A weak $\dagger$-bimodule over B is a triplet $(X, \dagger, {}_B\langle\langle\ ,\ \rangle\rangle)$ where X is a B-bimodule which is finitely generated and projective as a left B-module, $\dagger$ is a $\dagger$-structure on X and ${}_B\langle\langle\ ,\ \rangle\rangle : X \times X \to B$ is a sesquilinear form such that ${}_B\langle\langle xb, y\rangle\rangle = {}_B\langle\langle x, yb^*\rangle\rangle$for all $x, y \in X, b \in B$. If ${}_B\langle\langle\ ,\ \rangle\rangle$ is an inner product, then we say that X is a $\dagger$-bimodule.

Definition: A quintuple $(\Omega^1, d', \dagger, \psi, {}_B\langle\langle\ ,\ \rangle\rangle)$ is said to be a weak Hermitian differential structure on a $*$-algebra B if

(1) (Ω^1, d', ψ) is a weak second order differential structure, and (2) the triple $(\Omega^1, \dagger, {}_B\langle\langle\ ,\ \rangle\rangle)$ is a weak $\dagger$-bimodule satisfying the condition $d'(b^*) = -d'(b)^\dagger$ for all $b \in B$.

Definition: A left connection ∇' is said to be a Levi-Civita connection in the sense of [MR25] for $(\Omega^1, d', \dagger, \psi, {}_B\langle\langle\ ,\ \rangle\rangle)$ if ∇' is Hermitian with respect to the triplet $(\Omega^1, \dagger, {}_B\langle\langle\ ,\ \rangle\rangle)$ i.e,

$$d'({}_B\langle\langle e, f\rangle\rangle) = {}_{\Omega^1}\langle\langle \nabla'(e), f\rangle\rangle - {}_{\Omega^1}\langle\langle e, \nabla'(f)\rangle\rangle$$

holds), and if ∇' is torsionless in the sense of [MR25] for the weak second order differential structure (Ω^1, d', ψ).

Comparison of Levi-Civita connections ([BGG26])

Suppose that $(\Omega^\bullet, \wedge, d, s)$ is a quasi-tame $*$-differential calculus, $(g, (\ ,\ ))$ is a real metric on Ω^1 and (∇, σ) a $*$-preserving bimodule connection on Ω^1 such that σ is an isomorphism. Then the following statements are equivalent:

(i) (∇, σ) is a (unique) Levi-Civita connection for the metric $(g, (\ ,\ ))$,
(ii) $\sqrt{-1}\nabla$ is a (unique) Levi-Civita connection in the sense of [MR25] for the associated weak Hermitian differential structure.

Examples ([BGG26])

The Heckenberger–Kolb calculi $\Omega_q^\bullet(G/L_S)$ is a quasi-tame differential calculus and hence the Levi-Civita connection on $\Omega^1(G/L_S)$ is also a Levi-Civita connection in the sense of [MR25]. A similar statement holds for unitary 2-cocycle deformation of classical Levi-Civita connections.

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