# Short star products for quantum symmetric pairs

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3 August 2026

based on arXiv: 2603.06132
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# Star products

$$A = \bigoplus_{l=0}^{\infty} A_l \quad \text{graded } \mathbb{K}\text{-algebra}$$

**Note**: $A$ is filtered with $\tilde{\mathcal{F}}_n(A) = \bigoplus_{l=0}^{n} A_l$

**star product**: $* : A \otimes A \to A$ bilinear associative operation s.th

$(A, *, \tilde{\mathcal{F}}_*)$ is filtered algebra w $\mathrm{gr}_{\tilde{\mathcal{F}}}(A, *) = A$.

$*$ is **$\mathbb{O}$-equivariant** if $h * a = ha$, $a * h = ah$ for all $h \in A_0$, $a \in A$

$*$ is **$\mathbb{Z}_2$-equivariant** if $A_m * A_n \subseteq \bigoplus_{l=0}^{\lfloor \frac{m+n}{2} \rfloor} A_{m+n-2l}$

$*$ is **short** if $A_m * A_n \subseteq \bigoplus_{l=|m-n|}^{m+n} A_l$

(Beem, Peelaers, Rastelli 2017: "truncation condition"

Etingof, Stryker 2020: "short")

**Filtered quantization of $A$**: Filtered algebra $B$ with fixed isomorphism $\phi : \mathrm{gr}(B) \longrightarrow A$

**Example:** $\mathcal{O} = \mathcal{O}_q(SL_2)$, $U = U_q(sl_2) = \mathbb{K}\langle E, F, K^{\pm 1}\rangle / \sim$

$$B = \mathcal{O}^K = \{a \in \mathcal{O} \mid a \triangleleft K = a\} = \bigoplus_{n=0}^{\infty} L(2n)$$

$$A = \bigoplus_{n=0}^{\infty} L(2n) \text{ with Cartan multiplication}$$

$\rightsquigarrow$ $B$ is a $*$-product on $A$

$$L(2m) \otimes L(2n) = \bigoplus_{l=|m-n|}^{m+n} L(2l) \quad \rightsquigarrow \quad B \text{ is short.}$$

**Generalization:** $Y = G/K = \mathrm{spec}(B)$ : homogeneous, spherical affine $G$-variety

$$Y \xrightarrow{\text{Popov '87: contraction}} \mathrm{gr}(Y) = \mathrm{spec}(\underbrace{\mathrm{gr}(B)}_{=: A})$$ horospherical

i.e. stabilizers contain maximal unipotent subgroup

$\mathrm{gr}(Y) \supseteq G/S$ dense, open

$B$ is short $*$-product on $A$

$S$ : "horospherical subgroup" (Knop)

In our case : $Y$ symmetric, $\mathfrak{k} = \mathrm{Lie}(K)$, $\mathfrak{s} := \mathrm{Lie}(S)$

**Goal:** $(U_q'(\mathfrak{k}), U_q(\mathfrak{s}))$ is a short $*$-product.

## Symmetric pairs

$\mathfrak{g}$ semisimple $/\mathbb{C}$ , $\theta : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}$, $\theta^2 = \mathrm{id}$

$(\mathfrak{g}, \theta)$ symmetric Lie algebra , $\mathfrak{g} = \underset{+1}{\mathfrak{k}} \oplus \underset{-1}{\mathfrak{p}}$ $(\mathfrak{g}, \mathfrak{k})$ symmetric pair

## Satake diagrams

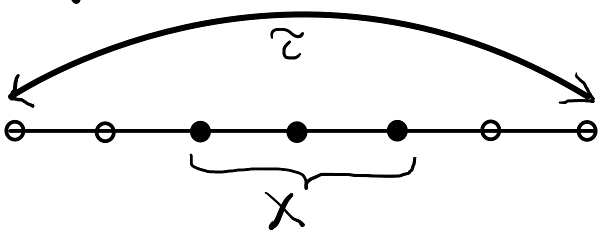
$(X, \tau)$ , $X \subseteq I$ , $\tau : I \to I$ diagram automorphism, i.e. $a_{\tau(i)\tau(j)} = a_{ij}$

s.th. 1) $\tau^2 = \mathrm{id}_I$

2) $\tau|_X = -w_X$ , i.e $w_X(\alpha_i) = -\alpha_{\tau(i)}$

3) $\forall\, j \in I \setminus X$ and $\tau(j) = j$ then $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$

↑ half sum of positive coroots

Examples: 

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**Notation:** $I_\circ = I \setminus X$

$(X,\tau)$ Satake diagram

$\omega : \mathfrak{g} \to \mathfrak{g}$ Chevalley involution

$\omega(e_i) = f_i,\ \omega(f_i) = e_i,\ \omega(h_i) = -h_i$

$\mathrm{Ad}(s_i) = \exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-f_i))\exp(\mathrm{ad}(e_i))$

$\rightsquigarrow$ Braid group action on $\mathfrak{g}$

$$(X,\tau) \longmapsto \Theta(X,\tau) := \mathrm{Ad}(\eta)\circ \mathrm{Ad}(w_X)\circ \tau\circ\omega \qquad \mathrm{Ad}(\eta)\big|_{\mathfrak{g}_\alpha} = \underbrace{\eta(\alpha)}_{\in\{\pm 1\}}\,\mathrm{id}\big|_{\mathfrak{g}_\alpha}$$

<u>Theorem</u> (Satake '62, Kac & S. Wang '92) : This is a bijection.

<u>Properties of $\Theta := \Theta(X,\tau)$</u>: $\Theta(\mathfrak{h}) = \mathfrak{h}$, $\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-w_X\tau(\alpha)}$, $\Theta|_{\mathfrak{g}_X} = \mathrm{id}_{\mathfrak{g}_X}$

<u>Generators of $\mathfrak{k}$</u>:

$$\left.\begin{array}{l} 1)\ \mathfrak{g}_X \\ 2)\ \mathfrak{h}^{\Theta} = \mathfrak{k}\cap\mathfrak{h} \end{array}\right\} =: \mathfrak{m}$$

3) $b_i := f_i + \Theta(f_i) = f_i + \eta(\alpha_{\tau(i)})\,\mathrm{Ad}(w_X)(e_{\tau(i)})\quad \forall i\in I_\circ$

# Quantum symmetric pairs

$\theta : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}$, $\theta^2 = \mathrm{id}$ involutive Lie alg. automorphism

$\rightsquigarrow$ $\mathfrak{g} = \underset{+1}{\mathfrak{k}} \oplus \underset{-1}{\mathfrak{p}}$ $\rightsquigarrow$ $U(\mathfrak{k}) \subseteq U(\mathfrak{g})$ Hopf subalgebra

Drinfeld-Jimbo: $\underbrace{U_q(\mathfrak{g})}_{=:U} = \underbrace{\mathbb{Q}(q)}_{=:\mathbb{K}} \langle E_i, F_i, K_h \mid i \in I, h \in Q^\vee \rangle /\sim$

$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$, $\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$, $\Delta(K_h) = K_h \otimes K_h$ $\left( K_i = K_{d_i h_i} \quad d_i = \frac{(\alpha_i, \alpha_i)}{2} \right)$

Problem: $U_q(\mathfrak{k}) \not\subseteq U_q(\mathfrak{g})$ not Hopf subalgebra (even if both are semisimple)

Definition (G. Letzter '99): For $\underline{c} = (c_i)_{i \in I \setminus X} \in (\mathbb{K}^*)^{I_0}$ let $B_{\underline{c}} \subseteq U$ be the subalgebra of $U = U_q(\mathfrak{g})$ generated by

$$\left.\begin{array}{l} 1)\ U_q(\mathfrak{g}_X) \\ 2)\ K_h \quad \text{with } \theta(h) = h \end{array}\right\} =: H_{X,\tau}$$

3) $B_i = F_i + c_i T_{w_X}(E_{\tau(i)}) K_i^{-1} \qquad \forall i \in I_0$

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$$3) \quad B_i = F_i + c_i T_{w_X}(E_{\tau(i)}) K_i^{-1} \qquad \forall i \in I_0$$

$$\boxed{\begin{array}{l} \underline{\text{Properties of } B_{\underline{c}} \text{ (Letzter '99)}} \\ 1) \ \Delta(B_{\underline{c}}) \subseteq B_{\underline{c}} \otimes U \quad \Delta B_i = B_i \otimes K_i^{-1} + 1 \otimes B_i - c_i q^{-(w_X \alpha_{\tau(i)}, \alpha_i)} (K_i^{-1} \otimes K_i^{-1}) \left( \Delta(T_{w_X}(E_{\tau(i)})) - T_{w_X}(E_{\tau(i)}) \otimes 1 \right) \\ 2) \ B_{\underline{c}} \xrightarrow{q \to 1} U(\mathfrak{k}) \qquad \text{for suitable choice of } \underline{c} \end{array}}$$

Filtration on $B_{\underline{c}}$: $\quad \deg(B_i) = 1 \quad \forall i \in I_0$

$$\deg(u) = 0 \quad \forall u \in H_{X,\tau}$$

associated graded? Set $A^-_{X,\tau} := U^- H_{X,\tau} =: U_q(\mathfrak{s})$ "quantum horospherical subalgebra"

$A^-_{X,\tau}$ is graded by $\deg(F_i) = 1 \ \forall i \in I_0$, $\deg(u) = 0 \ \forall u \in H_{X,\tau}$

**Lemma:** $\exists$ surjective algebra homomorphism $\rho : gr(B_c) \longrightarrow A^-_{X,\tau}$ such that
$$B_i \longmapsto F_i \quad \forall\, i \in I_0$$
$$u \longmapsto u \quad \forall\, u \in H_{X,\tau}$$

Let $I_\tau$ be a set of representatives of $\tau$-orbits in $I_0$

**Theorem** (Letzter '99, '04, K '14) : TFAE

1. $\rho$ is an isomorphism.
2. Multiplication: $B_{\mathbf{c}} \otimes \mathbb{K}\langle K_{\pm h_i} \mid i \in I_\tau \rangle \otimes R^+_X \longrightarrow U$ is a linear isomorphism. (quantum positive nilradical for $\mathfrak{p}^+_X$)
3. $B_{\mathbf{c}} \cap U^0 = \mathbb{K}\langle K_h \mid \Theta(h) = h \rangle$.
4. $U$ is a free left $B_{\mathbf{c}}$-module.
5. $c_i = c_{\tau(i)} \quad \forall\, i \in I_0$ with $(\alpha_i, w_X(\alpha_{\tau(i)})) = 0$.

From now on: choose $\mathbf{c} = (c_i)_{i \in I \setminus X}$ as in 5

$\rightsquigarrow B_{\mathbf{c}}$ is a filtered quantization of $A^-_{X,\tau}$.

# The Letzter map

Define $\quad U^{poly} := \mathbb{K}\langle F_i, E_iK_i^{-1}, K_i^{-1}, h \mid i \in I\setminus X, h \in H_{X,\tau}\rangle$

note $\; E_iK_i^{-1}F_i - q_i^2 F_i E_i K_i^{-1} = \dfrac{1-K_i^{-2}}{q_i - q_i^{-1}}$

$$U^{poly} \cong \underbrace{R_X^- \otimes H_{X,\tau}}_{=A^-_{X,\tau}\ (\text{horospherical subalgebra } U^-H_{X,\tau})} \otimes \mathbb{K}[K_i^{-1} \mid i \in I_\tau] \otimes R_X^+$$

where $R_X^- = U^- \cap T_{w_X}(U^-)$

$R_X^+ = S(U^+ \cap T_{w_X}(U^+))$

$$= A^-_{X,\tau} \oplus \mathcal{J}$$

$\mathcal{J}$: left ideal generated by $K_i^{-1}$ for $i \in I_\tau$ and $R^+_{X,+}$

$$\begin{tikzcd} B_c \arrow[r, hook] \arrow[dr, "\psi_B"'] & U^{poly} \arrow[d, "\psi\ \text{projection}"] \\ & A^-_{X,\tau} \end{tikzcd}$$

Letzter map

**Proposition** (K & Yakimov '21)

1) $\psi_B$ is a linear map of filtered vector spaces.

2) $\psi(ab) = \psi(a\,\psi(b)) \quad \forall\, a, b \in U^{poly}$

3) $\psi$ is $H_{X,\tau}$-bilinear

4) $\mathrm{gr}(\psi_B) = \rho$

(in particular, $\psi_B$ is a linear isomorphism)

For simplicity write: $A = \bar{A}_{X,\tau}$, $H = H_{X,\tau}$

$$
\begin{tikzcd}
B_{\underline{c}} \arrow[rr, hook] \arrow[dr, "\psi_B"'] & & U^{poly} \arrow[dl, "\psi"', "\text{projection}"] \\
& A &
\end{tikzcd}
$$

Letzter map

**Proposition** (K & Yakimov '21)

1) $\psi_B$ is a linear map of filtered vector spaces.
2) $\psi(ab) = \psi(a\,\psi(b)) \quad \forall\, a, b \in U^{poly}$
3) $\psi$ is $H$-bilinear
4) $\mathrm{gr}(\psi_B) = \rho$

$\rightsquigarrow \psi_B : B_{\underline{c}} \longrightarrow A$ is a quantization map for $\rho : \mathrm{gr}(B_{\underline{c}}) \longrightarrow A$.

$\rightsquigarrow \ast : A \otimes A \longrightarrow A$, $a \ast b = \psi_B(\psi_B^{-1}(a)\,\psi_B^{-1}(b))$ is a $\ast$-product on $A$.

3) $\rightsquigarrow$ $\ast$ is $\mathcal{O}$-equivariant.

$\boxed{\textbf{Theorem} \text{ (K \& Yakimov '26): } (A, \ast) \text{ is a short star product.}}$

# Main result

**Theorem** (K & Yakimov '26): $(A, *)$ is a short star product.

**Proof**: Need to show $$A_m * A_n \subseteq \bigoplus_{\ell=|m-n|}^{m+n} A_\ell \qquad (1)$$

- Define $\mathcal{W} := U^{poly}/\langle K_i^{-1} \mid i\in I_\tau\rangle \cong R_x^- \otimes H \otimes R_x^+$ (horospherical Heisenb. double)

**Lemma**: 1) $\mathcal{W}$ acts on $A$ via $w \triangleright a = \psi(wa)$

2) $A$ is a Fock space for $\mathcal{W}$, i.e cyclic and $R_x^+$: lowering operators, $R_x^-$: raising operators.

- $\mathcal{W}_m := \sum_{\ell=0} R^-_{x,m-\ell} H R^+_{x,\ell}$ (vector space grading)
- $\pi(B_\subseteq)$ is a graded subspace of $\mathcal{W}$
- $\wr$ is a graded linear map

$\rightsquigarrow$ $\pi(\psi_B^{-1}(a)) \in \mathcal{W}_m$ if $a \in A_m$.

$$\begin{tikzcd}
B_\subseteq \arrow[rr, hook] \arrow[dr, "\pi"] \arrow[ddr, "\psi_B"'] & & U^{poly} \arrow[dl, "\pi"'] \arrow[ddl, "\psi"] \\
& \mathcal{W} \arrow[d, "\wr"] & \\
& A &
\end{tikzcd}$$

# Main result

**Theorem** (K & Yakimov '26): $(A, *)$ is a short star product.

**Proof**: Need to show
$$A_m * A_n \subseteq \bigoplus_{\ell=|m-n|}^{m+n} A_\ell \qquad (1)$$

- Define $\mathcal{W} := U^{poly} / \langle K_i^{-1} \mid i \in I_\tau \rangle \cong R_x^- \otimes H \otimes R_x^+$ (horospherical Heisenb. double)
- $\mathcal{W}_m := \sum_{\ell=0} R^-_{x,m-\ell} H R^+_{x,\ell}$ (vector space grading)

$$\begin{tikzcd}
B_\leq \arrow[rr, hook] \arrow[dr, "\pi"] \arrow[ddr, "\psi_B"'] & & U^{poly} \arrow[dl, "\pi"'] \arrow[ddl, "\psi"] \\
& \mathcal{W} \arrow[d, "\iota", "\wr"'] & \\
& A &
\end{tikzcd}$$

- $\pi(B_\leq)$ is a graded subspace of $\mathcal{W}$
- $\iota$ is a graded linear map

$\rightsquigarrow \quad \pi(\psi_B^{-1}(a)) \in \mathcal{W}_m$ if $a \in A_m$.

- $a * b = \psi_B(\psi_B^{-1}(a)\psi_B^{-1}(b)) = \psi_B(\pi(\psi_B^{-1}(a))\, b) = \pi(\psi_B^{-1}(a)) \triangleright b$

$a \in A_m$, $b \in A_n$

$$\rightsquigarrow \quad a * b \in \bigoplus_{\ell=|m-n|}^{m+n} A_\ell$$

$\blacksquare$

## Reformulation

Define $\mu_i^L : A \to A$, $\mu_i^L(a) = F_i * a - F_i a$ $\quad \forall i \in I_0$

$\mu_i^R : A \to A$, $\mu_i^R(a) = a * F_i - a F_i$

As $A$ is generated in degrees 0 and 1, and $*$ is 0-equivariant, the maps $(\mu_i^L \mid i \in I_0)$ determine $*$ uniquely (similarly $\mu_i^R$).

> **Corollary:** For all $i \in I \setminus X$ the maps
> $$\mu_i^L, \mu_i^R : A \longrightarrow A$$
> are homogeneous of degree $-1$.

**Note:** The statement of the corollary is equivalent to the shortness of $*$ (if we know that $*$ is $\mathbb{Z}_2$-equivariant).

<u>Note</u>: It is easier to determine $\mu_i^L$ than to determine $\mu_i^R$.

e.g. $X = \emptyset \rightsquigarrow A = U^- U^0_\theta$, $B_i = F_i - c_i E_{\tau(i)} K_i^{-1}$

For $u = \psi(b) \in U^-$, $b \in B_c$ we have $[E_i, u] = \dfrac{K_i \partial_i^L(u) - \partial_i^R(u) K_i^{-1}}{q_i - q_i^{-1}}$

$$F_i * u = \psi_B(B_i\, \psi_B^{-1}(u)) = \psi\left((F_i - c_i E_{\tau(i)} K_i^{-1})\, u\right) \qquad (i \in I_0)$$

$$= \psi(F_i u) - c_i q^{(\alpha_i, \alpha_{\tau(i)})}\, \psi\left(K_i^{-1} [E_{\tau(i)}, u]\right)$$

$$= F_i u - c_i \frac{q^{(\alpha_i, \alpha_{\tau(i)})}}{q_i - q_i^{-1}} K_i^{-1} K_{\tau(i)}\, \partial^L_{\tau(i)}(u)$$

<u>General case</u> (K & Yakimov '21): For all $i \in I_0$, $u \in R_X^-$ we have

$$F_i * u = F_i u - c_i \frac{q^{(\alpha_i, w_X \alpha_{\tau(i)})}}{q_i - q_i^{-1}} K_i^{-1} K_{w_X \alpha_{\tau(i)}}\, T_{w_X} \circ \partial^L_{\tau(i)} \circ T_{w_X}^{-1}(u)$$

<u>Note</u>: Hence $(A, *)$ is $\mathbb{Z}/2$-graded, i.e. $A_m * A_n \subseteq \bigoplus_{\substack{\ell \leq m+n \\ \ell \equiv m+n \bmod 2}} A_\ell$

($\mathbb{Z}/2$-equivariant $*$-product)

# Determining $\mu_i^R$

**Proposition**: The maps $\mu_\ell^L, \mu_i^R : A \to A$ for $\ell, i \in I_0$ satisfy

(1) $\mu_i^R \circ \mu_\ell^L = \mu_\ell^L \circ \mu_i^R$

(2) $\mu_i^R \circ T_{w_x}^{-1} \circ \mu_\ell^L \circ T_{w_x} = T_{w_x}^{-1} \circ \mu_\ell^L \circ T_{w_x} \circ \mu_i^R$

**Proof**: (1) $(F_\ell * a) * F_i = F_\ell a F_i + \mu_\ell^L(a) F_i + \mu_i^R(F_\ell a) + \mu_i^R(\mu_\ell^L(a))$

$(F_\ell * (a * F_i)) = F_\ell a F_i + F_\ell(\mu_i^R(a)) + \mu_\ell^L(a F_i) + \mu_\ell^L(\mu_i^R(a))$

Now, for $a \in A_n$, compare component in $A_{n-2}$. $\blacksquare$

$\hookrightarrow$ $\mu_i^R$ satisfies a skew derivation property

$\hookrightarrow$ the same skew derivation property is satisfied by $\sigma\tilde{\tau}\, \mu_{\tilde{\tau}(i)}^L\, \sigma\tilde{\tau}$

$$\boxed{\textbf{Theorem (K \& Yakimov '26)}: \quad \mu_i^R = \sigma\tilde{\tau}\, \mu_{\tilde{\tau}(i)}^L\, \sigma\tilde{\tau} \quad \forall\, i \in I_0}$$

**Recall**: $\sigma : U \to U$ algebra anti-automorphism with $\sigma(E_i) = E_i$, $\sigma(F_i) = F_i$, $\sigma(K_i) = K_i^{-1}$

# Application 1: Bar involution for $\mathbf{B}_{\underline{c}}$ (recall $\mathbb{K} = \mathbb{Q}(q)$)

- $\overline{\phantom{u}} : U \longrightarrow U,\ u \longmapsto \overline{u}$ bar involution, $\mathbb{Q}$-linear alg. automorphism
  s.th $\overline{E_i} = E_i$, $\overline{F_i} = F_i$, $\overline{K_i} = K_i$, $\overline{q} = q^{-1}$ $\forall i \in I$.
- Would like to have (Stroppel 2012, Bao & Wang 2013)
  $\overline{\phantom{b}}^{B} : \mathbf{B}_{\underline{c}} \longrightarrow \mathbf{B}_{\underline{c}},\ b \longmapsto \overline{b}^{B}$ $\mathbb{Q}$-linear alg. automorphism
  s.th $\overline{B_i}^{B} = B_i$, $\overline{\phantom{b}}^{B}\big|_H = \overline{\phantom{b}}\big|_H$ (existence proved in full generality for KM in 2022; finite type in 2015)

<u>Note</u>: $\overline{\phantom{b}}$ does not map $\mathbf{B}_{\underline{c}}$ to itself.

We can use the $*$-product to construct the bar involution on $\mathbf{B}_{\underline{c}}$.

# Bar involution for $B_{\underline{c}}$ (cntd.)

**Theorem** (K & Yakimov '26): $\sigma\tau : (A,*) \longrightarrow (A,*)$ is an algebra anti-automorphism of the star product algebra $(A,*)$.

Proof: Suffices to check $\sigma\tau(F_\ell * a) = \sigma\tau(a) * \sigma\tau(F_\ell)$.

This follows from $\mu_i^R = \sigma\tau \circ \mu^L_{\tau(i)} \circ \sigma\tau$ $\square$

Recall: The Letzter map $\psi_B : B_c \longrightarrow (A,*)$ is an algebra isomorphism.

Corollary: $\exists\, \sigma_\tau : B_{\underline{c}} \to B_{\underline{c}}$ algebra anti-automorphism, s.th

$$\sigma_\tau(B_i) = B_{\tau(i)} \ \forall i \in I_0 \quad \text{and} \quad \sigma_\tau|_H = \sigma\circ\tau|_H .$$

Lemma (Bao & Wang '21): Assume $\overline{c_i} = (-1)^{2\alpha_i(\rho_X^\vee)} q^{(2\rho_X - \Theta(\alpha_i), \alpha_i)} c_{\tau(i)}$ $\forall i \in I_0$.

Then $\sigma\circ\tau\circ\overline{\phantom{x}} : U \to U$ satisfies $\sigma\circ\tau\circ\overline{\phantom{x}}(B_i) = B_{\tau(i)}$ $\forall i \in I_0$

and hence defines a $\mathbb{Q}$-algebra anti-automorphism $B_{\underline{c}} \longrightarrow B_{\underline{c}}$.

# Bar involution for $B_{\underline{c}}$ (cntd.)

<u>Corollary</u>: $\exists\, \sigma_\tau : B_{\underline{c}} \to B_{\underline{c}}$ algebra anti-automorphism, s.th

$$\sigma_\tau(B_i) = B_{\tau(i)} \;\forall i \in I_0 \quad \text{and} \quad \sigma_\tau\big|_H = \sigma \circ \tau\big|_H .$$

<u>Lemma</u> (Bao & Wang '21): Assume $\overline{c_i} = (-1)^{2\alpha_i(\rho_X^\vee)} q^{(2\rho_X - \Theta(\alpha_i), \alpha_i)} c_{\tau(i)} \;\forall i \in I_0$.

Then $\sigma \circ \tau \circ \overline{\phantom{x}} : U \to U$ satisfies $\sigma \circ \tau \circ \overline{\phantom{x}}(B_i) = B_{\tau(i)} \;\forall i \in I_0$

and hence defines a $\mathbb{Q}$-algebra anti-automorphism $B_{\underline{c}} \to B_{\underline{c}}$.

<u>Theorem</u>: Assume $\overline{c_i} = (-1)^{2\alpha_i(\rho_X^\vee)} q^{(2\rho_X - \Theta(\alpha_i), \alpha_i)} c_{\tau(i)} \;\forall i \in I_0$.

Then $\overline{\phantom{x}}^{B} := \sigma_\tau \circ \sigma \circ \tau \circ \overline{\phantom{x}} : B_{\underline{c}} \longrightarrow B_{\underline{c}}, \quad b \longmapsto \overline{b}^{B}$

is a $\mathbb{Q}$-algebra automorphism such that

$$\overline{B_i}^{B} = B_i \;\forall i \in I_0 \quad \text{and} \quad \overline{\phantom{x}}^{B}\big|_H = \overline{\phantom{x}}\big|_H .$$

# Application 2 : Bar involution for $B_{\underline{c}}$-modules

**Recall** : • $B_{\underline{c}} \subset U$ subalgebra such that $\Delta(B_{\underline{c}}) \subseteq B_{\underline{c}} \otimes U$.

• bar involutions : $\bar{\ } : U \longrightarrow U$ , $\bar{\ }^{B} : B_{\underline{c}} \to B_{\underline{c}}$ .

**Some terminology** (related to based modules) :

- A $U$-module $V$ has a "$U$-bar involution" if $\exists\, \mathbb{Q}$-linear map $\bar{\ }^{u} : V \to V$ s.th $\overline{xv}^{u} = \bar{x}\ \bar{v}^{u}$ $\forall x \in U, v \in V$
- A $B_{\underline{c}}$-module $M$ has a "$B$-bar involution" if $\exists\, \mathbb{Q}$-linear map $\bar{\ }^{B} : M \to M$ s.th $\overline{bm}^{B} = \bar{b}^{B}\ \bar{m}^{B}$ $\forall b \in B_{\underline{c}}, m \in M$
- Let $V, W$ be $U$-modules with $U$-bar involution
  $M$ be a $B_{\underline{c}}$-module with $B$-bar involution.

3 Problems :
1) Find $U$-bar involution on $V \otimes W$.
2) Find $B$-bar involution on $V$ (as a $B_{\underline{c}}$-module).
3) Find $B$-bar involution on $M \otimes V$.

**Problem 1:** Let $V, W$ be $U$-modules with $U$-bar involution. Find a $U$-bar involution on $V \otimes W$.

**Solution** (Lusztig) : Find $\Theta$ (acting on $V \otimes W$) s.th $\Delta(\bar{u})\Theta = \Theta(\bar{\ } \otimes \bar{\ })\Delta(u)$

Set $\overline{v \otimes w} = \Theta\, \bar{v} \otimes \bar{w} \quad \forall v \in V, w \in W$

then $\overline{u \cdot (v \otimes w)} = \Theta(\bar{\ } \otimes \bar{\ })\big(\Delta(u)\cdot(v \otimes w)\big) = \Theta(\bar{\ } \otimes \bar{\ })(\Delta(u))\cdot(\bar{v} \otimes \bar{w})$

$= \Delta(\bar{u})\,\Theta(\bar{v} \otimes \bar{w}) = \bar{u} \cdot (\overline{v \otimes w})$

Construction: $\Theta = \sum_{\mu \in Q^+} \Theta_\mu \quad , \quad \Theta_\mu \in U^-_{-\mu} \otimes U^+_\mu$ "quasi R-matrix"

**Problem 2**: Find a B-bar involution on $V$

**Solution** (Bao & Wang '13): Find $\Upsilon$ s.th $\Upsilon \overline{b} = \overline{b}^B \Upsilon \quad \forall b \in B_c$

Set $\overline{v}^B = \Upsilon \overline{v}^u \quad \forall v \in V$

then $\overline{bv}^B = \Upsilon \overline{bv}^u = \Upsilon \overline{b} \, \overline{v}^u = \overline{b}^B \Upsilon \overline{v}^u = \overline{b}^B \overline{v}^B$

Construction: $\Upsilon = \sum_{\mu \in Q^+} \Upsilon_\mu$ with $\Upsilon_\mu \in U_\mu^+$

Existence (Bao & Wang '18, Balagović & K '19): Induction, assuming existence of bar involution $\overline{\phantom{x}}^B$.

(Appel & Vlaar '22, Wang & Zhang '23): bar involution free reformulation

$\exists \, \sigma_\tau : B_c \to B_c$ algebra anti-automorphism s.th

$\sigma_\tau(B_i) = B_{\tau(i)}, \quad \sigma_\tau|_{\mathcal{H}} = \sigma \circ \tau|_{\mathcal{H}}$

where $\sigma : U \to U$ is algebra anti-autom given by $\sigma(E_i) = E_i$, $\sigma(F_i) = F_i$, $\sigma(K_i) = K_i^{-1}$

then $\Upsilon$ determined by: $\sigma_\tau(b)\Upsilon = \Upsilon \sigma \circ \tau(b)$

$\Upsilon$ is called "quasi K-matrix"

**Problem 3:** Find B-bar involution on $M \otimes V$

**Solution** (Bao & Wang '13): Set $\Theta^B := \Delta(\mathcal{X}) \Theta (\mathcal{X}^{-1} \otimes 1)$

$(*)$ then $\Delta(\overline{b}^B)\Theta^B = \Theta^B(\overline{\phantom{x}}^B \otimes \overline{\phantom{x}})(\Delta(b)) \quad \forall\, b \in B_{\underline{c}}$.

Can show: $\Theta^B = \sum_{\mu \in Q^+} \Theta^B_\mu$ with $\Theta^B_\mu \in B_{\underline{c}} \otimes U^+_\mu$

Set $\overline{m \otimes v}^B := \Theta^B\, \overline{m}^B \otimes \overline{v}^u$

Bar involution free version of intertwiner property $(*)$:

$$\Delta(\sigma_\tau(b))\,\Theta^B = \Theta^B(\sigma_\tau \otimes \sigma\tau)(\Delta(b)) \quad \forall\, b \in B_{\underline{c}}$$

$\Theta^B$ is called "tensor quasi K-matrix"

**Problem:** Find an 'explicit' formula for $\Theta^B$.

# The tensor quasi K-matrix

Recall the quasi R-matrix $\Theta$ satisfies $\Delta(\bar{u})\,\Theta = \Theta\cdot(\bar{\ }\otimes\bar{\ })(\Delta(u))$.

For $u = F_i$ we get $(F_i \otimes K_i^{-1} + 1\otimes F_i)\,\Theta = \Theta\,(F_i\otimes K_i + 1\otimes F_i)$

Rewrite this in terms of $*$ : (write formally $\Theta = \Theta_1 \otimes \Theta_2$)

$$(F_i * \Theta_1)\otimes K_i^{-1}\Theta_2 - \mu_i^L(\Theta_1)\otimes K_i^{-1}\Theta_2 + (1\otimes F_i)\,\Theta$$

$$= (\Theta_1 * F_i)\otimes \Theta_2 K_i - \mu_i^R(\Theta_1)\otimes \Theta_2 K_i + \Theta(1\otimes F_i)$$

Apply $\psi_B^{-1}\otimes \mathrm{id}$ to get the following :

$\boxed{\begin{array}{l}\underline{\text{Theorem}}\ (\text{K \& Yakimov '26}):\ \text{The element } \Theta^B := (\psi^{-1}\otimes 1)(\Theta) \\ \quad\text{satisfies the relation} \\ \qquad\qquad \Delta(\sigma_\tau(b))\,\Theta^B = \Theta^B(\sigma_\tau\otimes\sigma\tau)(\Delta(b)) \qquad \forall b\in B_c .\end{array}}$

$\rightsquigarrow$ $\boxed{\text{The tensor quasi K-matrix is } (\psi_B^{-1}\otimes 1)(\Theta).}$