Constructing semisimple Hopf algebras and non-inner faithful actions

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Goal

Construct new examples of semisimple Hopf algebras and their actions.

General assumption: $\mathrm{char}(\mathbb{K}) = 0$, $\overline{\mathbb{K}} = \mathbb{K}$

... Side Quests

Question

Let H be a semisimple Hopf algebra acting on A. Let

$$A^H = \{a \in A : \forall h \in H \; h \cdot a = \epsilon(h)a\}.$$

Suppose A has **no non-zero nilpotent** *H-stable ideals.*

1. *If A is non-unital, then $A^H \neq \{0\}$?*
2. *If A is unital and I an H-stable ideal of A, then $I \cap A^H \neq \{0\}$?*
3. *If A is unital, does $A \# H$ has no non-zero nilpotent ideals?*

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Starting Point ...

Theorem (Etingof-Walton 2013)

Any action of a semisimple Hopf algebra on a commutative domain factors through a group algebra action.

An action of H on A factors through a quotient Hopf algebra H/I if $I \cdot A = 0$.

☞ *Construct examples of semisimple Hopf algebras on non-commutative domains that do not factor though group actions.*

Extensions of Etingof-Walton's Theorem

Theorem (Pansera-L, 2015)

Any action of a semisimple Hopf algebra on

1. $A = U(\mathfrak{g})$, *for any finite dimensional Lie algebra* $\mathfrak{g}$,
2. $A = \mathbb{C}[x_0][x_1, \delta_1][x_2, \delta_2] \cdots [x_m, \delta_m]$.
3. $A = A_n(\mathbb{C})$ *(Cuadra-Etingof-Walton, 2014)*

factors through a group algebra.

Kac-Paljutkin algebra

Definition (Kac-Paljutkin, 1966)

$H_8 = \mathbb{K}\langle x, y, z\rangle$, with x, y commuting group-likes of order 2 and

$$\begin{aligned} zx &= yz, \qquad zy = xz, \\ z^2 &= \frac{1}{2}(1 + x + y - xy), \\ \Delta(z) &= \frac{1}{2}(1\otimes 1 + x\otimes 1 + 1\otimes y - x\otimes y)(z\otimes z). \end{aligned}$$

The antipode is defined by $S(z) = z$, $S(x) = x$, $S(y) = y$.

H_8 has basis $\{1, x, y, xy, z, xz, yz, xyz\}$.

A non-trivial action

$$H_8 = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2][z;\sigma]/\langle z^2 - \tfrac{1}{2}(1+x+y-xy)\rangle, \qquad \sigma : x \leftrightarrow y$$

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Example (Kirkman-Kuzmanovich-Zhang, 2009)

H_8 acts homogeneously on $A = \mathbb{C}_\imath[u,v] = \mathbb{C}\langle u, v : vu = \imath uv\rangle$ via

$$X = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note that $z \cdot (uv) = -vu \neq vu = (z\cdot u)(z \cdot v)$.

The action of H_8 on $\mathbb{C}_q[u,v]$ is **inner faithful** in the sense that it does not factor through a quotient Hopf algebra.

Kac-Paljutkin via matched pair of groups

A matched pair (G, F): actions $\triangleright : G \times F \to F$ and $\triangleleft : G \times F \to G$ such that $G \times F$ is a group with product

$$(a, x)(b, y) = (a(x \triangleright b), (x \triangleleft b)y).$$

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$\mathrm{Opext}(\mathbb{K}[F], \mathbb{K}[G]^*) =$ (Hopf algebra extensions of $\mathbb{K}[F]$ by $\mathbb{K}[G]^*$)$/ \sim$

$$\mathbb{K}[G]^* \hookrightarrow H \twoheadrightarrow \mathbb{K}[F]$$

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For $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle x, y \rangle$ and $F = \mathbb{Z}_2 = \langle z \rangle$ acting on G by

$$x^a y^b \triangleleft z := x^b y^a,$$

one has $H_8 \in \mathrm{Opext}(\mathbb{K}[\mathbb{Z}_2], \mathbb{K}[\mathbb{Z}_2 \times \mathbb{Z}_2]^*)$.

Kac-Paljutkin via skew polynomial rings

$$R = \mathbb{K}[\mathbb{Z}_2 \times \mathbb{Z}_2] = \mathbb{K}\langle x, y : xy = yx, x^2 = 1 = y^2\rangle$$

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$\Delta(z) = J(z \otimes z)$ and $\epsilon(z) = 1$, where

$$J = \frac{1}{2}(1 \otimes 1 + x \otimes 1 + 1 \otimes y - x \otimes y) \qquad \text{and} \qquad t = \mu_R(J).$$

Twisted homomorphisms

Definition

A Drinfeld twist of a bialgebra H is an invertible element $J \in H \otimes H$ with

1. $(\Delta \otimes \mathrm{id})(J)(J \otimes 1) = (\mathrm{id} \otimes \Delta)(J)(1 \otimes J)$
2. $(\epsilon \otimes \mathrm{id})(J) = 1 = (\mathrm{id} \otimes \epsilon)(J)$

If J is a twist, then

$$\Delta^J(h) := J\Delta(h)J^{-1}$$

defines another bialgebra structure on H, denoted by H^J.

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For a bialgebra R, $\sigma \in \mathrm{Aut}(R)$ and $J \in R \otimes R$ invertible, the following statements are equivalent:

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For a bialgebra R, $\sigma \in \mathrm{Aut}(R)$ and $J \in R \otimes R$ invertible, the following statements are equivalent:

1. *$R[z;\sigma]$ is a bialgebra extension of R and*

$$\Delta(z) = J(z \otimes z) \qquad \text{and} \qquad \epsilon(z) = 1.$$

2. *(σ, J) is a twisted homomorphism in the sense of A.Davydov, i.e.*
   - *J is a twist and*
   - *$\sigma : R \to R^J$ is a bialgebra automorphism.*

Skew Monoid Algebra

Let $\alpha : M \to \mathrm{End}_{\mathbb{K}}(R)$ be a monoid homomorphism, M monoid, R $\mathbb{K}$-algebra.

Definition

$R\#M$ is the free left R-module with basis the elements of M and multiplication

$$(r\#x)(s\#y) := r\,\alpha_x(s)\#xy.$$

Example

For $\sigma \in \mathrm{Aut}(R)$, $M = \{z^i : i \geq 0\}$ and $\alpha(z^i) = \sigma^i$, one gets $R[z;\sigma] = R\#M$.

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Twisted Monoid Action

Theorem

Suppose $\alpha : M \to \mathrm{Aut}(R)$ *is a homomorphism of monoids,* R *a bialgebra,* $M = \langle z_1, \ldots, z_m \rangle$ *a free monoid and central invertible* $J_1, \ldots, J_m \in R \otimes R$*. TFAE:*

1. $R\#M$ *is a bialgebra extension of* R *and*
$$\Delta(z_i) = J_i(z_i \otimes z_i) \qquad \text{and} \qquad \epsilon(z_i) = 1.$$
2. $(\alpha(z_1), J_1), \ldots, (\alpha(z_m), J_m)$ *are twisted homomorphisms.*

Generalization: substitute $\mathbb{Z}_2$-action by Σ_m-action

Let B be a Hopf algebra, e.g. $B = \mathbb{K}[\mathbb{Z}_n]$.

- instead of $\Sigma_2 = \mathbb{Z}_2 \curvearrowright \mathbb{K}[\mathbb{Z}_2 \times \mathbb{Z}_2]$,

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- Let J be a twist for B, e.g. $J = \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x^i \otimes x^j$ for $\mathbb{K}[\mathbb{Z}_n]$.

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- How to get compatible twists J_σ for $B^{\otimes m}$, $\sigma \in \Sigma_m$?

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- Use $\{J_\sigma : \sigma \in X\}$ to turn the skew monoid algebra $B^{\otimes m} \# M$, with M freely generated by some $X \subseteq \Sigma_m$, into a bialgebra.

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- Use $\{J_\sigma : \sigma \in X\}$ to turn the skew monoid algebra $B^{\otimes m} \# M$, with M freely generated by some $X \subseteq \Sigma_m$, into a bialgebra.
- Find a semisimple Hopf quotient $(B^{\otimes m} \# M)/I \in \mathrm{Opext}(\mathbb{K}[\Sigma_m], (B^{\otimes m})^*)$.

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Steps - strong twists

Let J be a twist for a bialgebra B.

Definition

J is called a *strong twist* if $(e_1^2 \otimes e_2^2)(J)$ is a twist for $B \otimes B$.

Theorem

If J is a central strong twist of B and $M = \langle z_1, \ldots, z_{m-1} \rangle$ a free monoid of rank $m-1$, then z_k acts on $B^{\otimes m}$ via $s_k = (k \;\; k{+}1)$ and $B^{\otimes m} \# M$ is a bialgebra extension of $B^{\otimes}$ with

$$\Delta(z_k) = J_{s_k}(z_k \otimes z_k), \qquad J_{s_k} = (e_k^m \otimes e_{k+1}^m)(J).$$

$$e_i^m : B \otimes B^{\otimes m} \qquad b \mapsto 1 \otimes \cdots \otimes b \otimes \cdots \otimes 1.$$

General construction

Theorem

Let J be a central strong twist for B and $M = \langle z_1, \dots, z_{m-1} \rangle$ the free monoid acting on $B^{\otimes m}$ via $s_k = (k\ \ k{+}1)$ and J_{s_k} as above. Let I be the ideal of $R\#M$ generated by

$$z_i^2 - \mu(J_{s_i}), \qquad z_i z_{i+1} z_i - z_{i+1} z_i z_{i+1}, \qquad z_i z_j - z_j z_i, \qquad |i-j| > 1.$$

Suppose B is a Hopf algebra with antipode S, such that

- $\Delta_{B\otimes B}(J) = (e_1^2 \otimes e_2^2)(J)(e_2^2 \otimes e_1^2)(J)(J \otimes J)$,
- $(\mathrm{id} \otimes S)(J) = J^{-1} = (S \otimes \mathrm{id})(J)$,
- $(S \otimes S)(J) = J$.

Then $H = (B^{\otimes m}\#M)/I$ is a Hopf algebra and $H \simeq B^{\otimes m}\#_\gamma \Sigma_m$.
If B is semisimple, then H is semisimple.

Generalized Kac-Paljutkin algebras

Theorem ($n, m \geq 2$, q an nth root of unity)

$\exists$ semisimple Hopf algebra $H_{n,m}$ of dimension $n^m m!$, generated by elements $x_1, \ldots, x_m, z_1, \ldots, z_{m-1}$, s.t.

$$x_i^n = 1, \qquad x_i x_j = x_i x_j, \qquad z_k x_i = x_{\sigma_k(i)} z_k$$

$$z_k z_{k+1} z_k = z_{k+1} z_k z_{k+1}, \qquad z_k z_l = z_l z_k, \ |k-l| > 1, \qquad z_k^2 = t_k$$

$$\Delta(z_k) = J_{\sigma_k}(z_k \otimes z_k), \quad J_{\sigma_k} = \frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x_k^i \otimes x_{k+1}^j, \quad t_k = \mu(J_{\sigma_k}).$$

$\sigma_k = (k \ \ k{+}1)$. $\qquad H_{n,m} \in \mathrm{Opext}(\mathbb{K}[\Sigma_m], \mathbb{K}[\mathbb{Z}_n^m])$.

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Generalized Kac-Paljutkin algebras, m=2,3

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$m = 3$: $H_{n,3}$ is generated by x_1, x_2, x_3, z_1, z_2 subject to

$$z_1x_1 = x_2z_1, \quad z_1x_2 = x_1z_1, \quad z_1x_3 = x_3z_1$$

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here $z_1 = \overline{\sigma_1}$, $z_2 = \overline{\sigma_2}$, with $\sigma_1 = (12)$, $\sigma_2 = (23)$ in Σ_3.

While the x_i are group-like, we have

$$\Delta(z_1) = \left(\frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x_1^i \otimes x_2^j \right) (z_1 \otimes z_1),$$

$$\Delta(z_2) = \left(\frac{1}{n} \sum_{i,j=0}^{n-1} q^{-ij} x_2^i \otimes x_3^j \right) (z_2 \otimes z_2).$$

$H_{n,3}$ has dimension $6n^3$ and basis $\left\{ x_1^i x_2^j x_3^k \overline{\sigma} : 0 \leq i, j, l < n, \sigma \in \Sigma_3 \right\}$.

$H_{n,3}$ is a crossed product $\mathbb{K}[\mathbb{Z}_n^m]\#_\gamma\Sigma_m$ where the 2-cocycle γ is determined by the multiplication in $H_{n,3}$ and given by $\gamma(\mathrm{id},\sigma)=\gamma(\sigma,\mathrm{id})=1$ and the following table.

γ	σ_1	σ_2	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_1$
σ_1	t_1	1	t_1	1	t_1
σ_2	1	t_2	1	t_2	t_2
$\sigma_1\sigma_2$	1	t_3	t_1	t_1t_3	t_1t_3
$\sigma_2\sigma_1$	t_3	1	t_2t_3	t_2	t_2t_3
$\sigma_1\sigma_2\sigma_1$	t_2	t_1	t_2t_3	t_1t_3	$t_1t_2t_3$

$$t_1=\frac{1}{n}\sum_{i,j=0}^{n-1}q^{-ij}x_1^ix_2^j,\qquad t_2=\frac{1}{n}\sum_{i,j=0}^{n-1}q^{-ij}x_2^ix_3^j,\qquad t_3=\frac{1}{n}\sum_{i,j=0}^{n-1}q^{-ij}x_1^ix_3^j.$$

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m-dimensional irreducible representations for $H_{n,m}$

Let $p = \sqrt{q}$ and $0 \leq a, b < n$. Then $H_{n,m}$ acts on $V_{a,b} = \bigoplus_{i=1}^{m} \mathbb{K}u_i$ as

$$X_i = \begin{bmatrix} q^b & & & & \\ & \ddots & & & \\ & & q^a & & \\ & & & \ddots & \\ & & & & q^b \end{bmatrix} \qquad Z_k = \begin{bmatrix} p^{b^2} & & & & \\ & \ddots & & & \\ & & \begin{matrix} 0 & 1 \\ q^{ab} & 0 \end{matrix} & & \\ & & & \ddots & \\ & & & & p^{b^2} \end{bmatrix}$$

$V_{a,b}$ is a simple left $H_{n,m}$-module, if $a \neq b \pmod{n}$.

Theorem

There exists numbers (r_{ij}) *for* $1 \leq i,j \leq m$, *such that* $H_{n,m}$ *acts on*

$$A_{ab} = \frac{\mathbb{K}\langle u_1, \ldots, u_m \rangle}{\langle u_i u_j - r_{ij} u_j u_i : i,j \rangle}, \tag{1}$$

such that $V_{a,b} = \mathbb{K}u_1 \oplus \cdots \oplus \mathbb{K}u_m$ *as* $H_{n,m}$*-submodule.*

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$$r_{i,j} = q^{(b^2-ab)(j-i-1)} p^{b^2-a^2}, \qquad i < j$$

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Consider the subalgebra H of $H_{n,3}$ generated by $R = \mathbb{K}[\mathbb{Z}_n^{\otimes 3}]$ and $\theta = z_1 z_2$.

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1. $H \simeq \dfrac{R[\theta, \overline{\sigma}]}{\langle \theta^3 - t_1 t_2 t_3 \rangle}$ *is a semisimple Hopf algebra of dimension $3n^3$, where* $\overline{\sigma}(x_i) = x_{\sigma(i)}$, *with* $\sigma = (123)$ *and*

$$\Delta(\theta) = \left(\frac{1}{n^2} \sum q^{-ij-kl} x_1^i x_2^k \otimes x_2^i x_3^l \right) (\theta \otimes \theta).$$

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3. *$A^H = \mathbb{K}[u_1^n, u_2^n, u_3^n]^{A_3}$ is the ring of cyclic polynomials.*

Theorem (Halbig, 2025)

$H_{n,m}$ is isomorphic as an algebra to the group algebra $\Bbbk[\mathbb{Z}_n \wr \Sigma_m]$.

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$\mathbb{Z}_n \wr \Sigma_m$ is a wreath product, i.e. as a set equal to $\mathbb{Z}_n^m \times \Sigma_m$ subject to

$$(a_1, \ldots, a_m, \sigma)(b_1, \ldots, b_m, \tau) = (a_1 b_\sigma(1), \ldots, a_m b_\sigma(m), \sigma\tau)$$

$\mathbb{Z}_n \wr \Sigma_m$ is the *generalized symmetric group* (W.Specht, Berlin, 1931).

Theorem (Halbig, 2025)

$H_{n,m}$ is isomorphic as an algebra to the group algebra $\Bbbk[\mathbb{Z}_n \wr \Sigma_m]$.

$\mathbb{Z}_n \wr \Sigma_m$ is a wreath product, i.e. as a set equal to $\mathbb{Z}_n^m \times \Sigma_m$ subject to

$$(a_1, \ldots, a_m, \sigma)(b_1, \ldots, b_m, \tau) = (a_1 b_\sigma(1), \ldots, a_m b_\sigma(m), \sigma\tau)$$

$\mathbb{Z}_n \wr \Sigma_m$ is the *generalized symmetric group* (W.Specht, Berlin, 1931).

$$z_k \mapsto y_k z_k, \qquad y_k = \sum_{\lambda \in \mathbb{Z}_n^m} p^{-\lambda_k \lambda_{k+1}} \Lambda_\lambda, \qquad \Lambda_\lambda = \frac{1}{n^m} \sum_{\eta \in \mathbb{Z}_n^m} q^{\lambda \cdot \eta} \mathbf{x}^\eta.$$

Corollary (Irreducible Representations of $H_{n,m}$)

$$\mathrm{Irr}(H_{n,m}) \cong \{\beta = (\lambda_1 \vdash l_1, \dots, \lambda_n \vdash l_n) : l_1 + \cdots + l_n = m.\}.$$

$$|\mathrm{Irr}(H_{n,m})| = \sum_{\substack{(l_1,\dots,l_n) \\ l_1+\cdots+l_n=m}} p(l_1)\cdots p(l_n), \quad \textit{where } p \textit{ denotes the partition function.}$$

$$\dim(V_\beta) = \frac{m!}{l_1! \cdot \cdots \cdot l_n!} f_0 \cdot \cdots \cdot f_{n-1}, \quad \textit{where } f_i = |\{\textit{std Young tabeaux shape } \lambda_i\}|$$

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$$H_{2,3} = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2]\#_\gamma \Sigma_3 \simeq \mathbb{C}[\mathbb{Z}_2 \wr \Sigma_3],$$ generators x_1, x_2, x_3, z_1, z_2

β	e_β	*dimension*
$((3),*)$	$\frac{1}{6}\Lambda_{(0,0,0)}(1+z_1+z_2+z_1z_2+z_2z_1+z_1z_2z_1)$	1
$((2,1),*)$	$\frac{1}{3}\Lambda_{(0,0,0)}(1+z_1)(1-z_1z_2z_1)$	2
$((1,1,1),*)$	$\frac{1}{6}\Lambda_{(0,0,0)}(1-z_1-z_2+z_1z_2+z_2z_1-z_1z_2z_1)$	1
$((2),(1))$	$\frac{1}{2}\Lambda_{(0,0,1)}(1+z_1)$	3
$((1,1),(1))$	$\frac{1}{2}\Lambda_{(0,0,1)}(1-z_1)$	3
$(*,(3))$	$\frac{1}{6}\Lambda_{(1,1,1)}(1+z_1+z_2+z_1z_2+z_2z_1+z_1z_2z_1)$	1
$(*,(2,1))$	$\frac{1}{3}\Lambda_{(1,1,1)}(1+z_1)(1-z_1z_2z_1)$	2
$(*,(1,1,1))$	$\frac{1}{6}\Lambda_{(1,1,1)}(1-z_1-z_2+z_1z_2+z_2z_1-z_1z_2z_1)$	1
$((1),(2))$	$\frac{1}{2}\Lambda_{(1,1,0)}(1+z_2)$	3
$((1),(1,1))$	$\frac{1}{2}\Lambda_{(1,1,0)}(1-z_2)$	3

$$\Lambda_{(a,b,c)} = \frac{1}{8}\sum_{(i,j,k)\in\mathbb{Z}_2^3}(-1)^{ai+bj+ck}x_1^i x_2^j x_3^k$$

Thank you for your attention!!!