Hopf algebroids and rings of differential operators

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Notation

- k field of characteristic zero
- A k-algebra
- $A^e := A \otimes A^{\mathrm{op}}$
- We identify A^e-**Mod** with the category of A-bimodules.

Hopf algebroids

Definition

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The map η induces a forgetful functor

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A **(left) bialgebroid** over A consists of an A^e-ring H together with a monoidal structure $(H\text{-}\mathbf{Mod}, \boxtimes, \mathbf{1})$ such that

$$\mathcal{U}(M \boxtimes N) = \mathcal{U}(M) \otimes_A \mathcal{U}(N), \qquad \mathcal{U}(\mathbf{1}) = A.$$

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A **(left) Hopf algebroid** is a bialgebroid H for which there exists a (left) inner hom $[\ ,\ ]_\ell$ on H-**Mod** such that

$$\mathcal{U}([M,N]_\ell) = \mathrm{Hom}_A(\mathcal{U}(M), \mathcal{U}(N)).$$

Structure maps

Structure maps

Proposition (Sweedler 1974, Takeuchi 1977, Schauenburg 1998)

A (left) bialgebroid structure on H is equivalent to an A-coring structure

$$\Delta\colon H \to H \otimes_A H, \qquad \varepsilon\colon H \to A,$$

on the left A^e-module ${}_{A^e}H$ such that Δ and ε satisfy some compatibility conditions with the multiplication on H.

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A (left) Hopf algebroid structure on H is equivalent to the invertibility of the (left) Galois map

$$\delta\colon H \otimes_{A^{\mathrm{op}}} H \to H \otimes_A H, \quad h \otimes_{A^{\mathrm{op}}} g \mapsto h_{(1)} \otimes_A h_{(2)} g.$$

where $h_{(1)} \otimes_A h_{(2)} := \Delta(h)$.

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where $h_{(1)} \otimes_A h_{(2)} := \Delta(h)$.

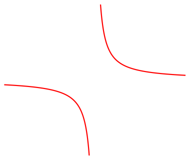
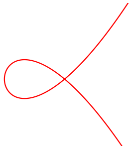
Beware: A Hopf algebroid does not admit an antipode in general.

The ring of differential operators: the commutative case

Main question

- We start with a **commutative algebra** A.
- We associate to this algebra its **ring of differential operators** $\mathscr{D}_A$.
- **Well-known:** If A is **smooth**, then $\mathscr{D}_A$ is a Hopf algebroid over A.
- **Question:** For which algebras A is $\mathscr{D}_A$ a Hopf algebroid?

Examples and counterexamples

Geometric object	A	$\mathscr{D}_A$	Hopf algebroid?
	$k[t, t^{-1}]$	localized Weyl algebra $U_K(\mathrm{Der}_k(K))$	Yes
	$k[t^2, t^3]$	generated by t^2, t^3 and some higher order operators	Yes
	$k + (t^2 - 1)k[t]$	$k + (t^2 - 1)\mathscr{D}_{k[t]}$	No

Differential operators

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Definition (Grothendieck 1966)

The **ring of k-linear differential operators** on A is the filtered subalgebra of $\mathrm{End}_k(A)$

$$\mathscr{D}_A := \bigcup_{n \geqslant 0} \mathscr{D}_A^n$$

where $\mathscr{D}_A^0 := \{\lambda_a \mid a \in A\} \cong A,$
$\mathscr{D}_A^n := \{D \in \mathrm{End}_k(A) \mid \forall a \in A : [D, \lambda_a] \in \mathscr{D}_A^{n-1}\}, \qquad n \geqslant 1.$

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For $n = 1$, we have $\mathscr{D}_A^1 = A \oplus \mathrm{Der}_k(A)$, where

$$\mathrm{Der}_k(A) = \{D \in \mathrm{End}_k(A) \mid D(ab) = aD(b) + D(a)b \, \forall a, b \in A\}.$$

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- Most difficult part: find the comultiplication map

$$\Delta_A \colon \mathscr{D}_A \to \mathscr{D}_A \otimes_A \mathscr{D}_A.$$

Smooth algebras

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Its **universal enveloping algebra** $\mathscr{U}_A(\mathrm{Der}_k(A))$ admits a Hopf algebroid structure:

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\begin{aligned}
\Delta(a) &= a \otimes_A 1, & \Delta(D) &= 1 \otimes_A D + D \otimes_A 1,\\
\varepsilon(a) &= a, & \varepsilon(D) &= 0,\\
\beta^{-1}(a \otimes_A 1) &= a \otimes_A 1, & \beta^{-1}(1 \otimes_A D) &= D \otimes_A 1 - 1 \otimes_A D,
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Theorem (Grothendieck 1966, Sweedler 1974, Narváez-Macarro 2009)

If $\mathrm{Der}_k(A)$ *is finitely generated projective over* A*, then* $\mathscr{U}_A(\mathrm{Der}_k(A)) \cong \mathscr{D}_A$.

How to compute $\mathscr{D}_A$ in general?

Here are some methods for the case when A is the coordinate ring of an affine variety.

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Quotients

Suppose $A = R/I$, where R is smooth over k. Then

$$\mathscr{D}_A \cong \{D \in \mathscr{D}_R \mid D(I) \subseteq I\}/\{D \in \mathscr{D}_R | D(R) \subseteq I\}.$$

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Localizations

Suppose $A \subseteq K$, where $K := S^{-1}A$ is a localization of A. Then we have

$$\mathscr{D}_K \cong K \otimes_A \mathscr{D}_A \quad \text{as a left } K\text{-module}$$

and

$$\mathscr{D}_A \cong \{D \in \mathscr{D}_K \mid D(A) \subseteq A\} \subseteq \mathscr{D}_K.$$

Descent technique

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Assume we have

- $A \subseteq K$ be commutative algebras
- H be a Hopf algebroid over K whose source and target maps are equal
- $H(A) := \{h \in H \mid \varepsilon(ha) \in A \, \forall a \in A\}$

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$$\begin{tikzcd} H \arrow[r, "\Delta"] & H \otimes_K H \\ H(A) \arrow[u, hook] & H(A) \otimes_A H(A) \arrow[u] \end{tikzcd}$$

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\end{tikzcd}$$

Question: When can we "restrict" the comultiplication of H to $H(A)$?

Local projectivity

Definition (Vercruysse 2008)

$H(A)$ is $R(A)$-**locally projective**, if for all $h \in H(A)$, there exist $a_1, \ldots, a_n \in A$ and $h_1, \ldots, h_n \in H(A)$ such that

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Theorem (Krähmer–M. 2026)

Suppose that

1. *$H(A)$ is $R(A)$-locally projective, and*
2. *the canonical map $K \otimes_A H(A) \to H$, $x \otimes h \mapsto xh$ is surjective.*

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This theorem can be applied directly to the setting where $A \subseteq K = S^{-1}A$, $H = \mathscr{D}_K$ and $H(A) = \mathscr{D}_A$.

Examples: Monomial Curves

- $\mathcal{A}$ **numerical semigroup**, i.e. a submonoid of $(\mathbb{N}, +, 0)$ such that $\mathbb{N} \backslash \mathcal{A}$ is finite
- $A := k[\mathcal{A}] = k[t^a \mid a \in \mathcal{A}]$
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Theorem (Krähmer–M. 2026)

1. *$\mathscr{D}_A$ is $R(A)$-locally projective, hence it is a Hopf algebroid over A.*
2. *$\mathscr{D}_A$ admits an antipode if $\mathcal{A}$ is* **symmetric**.
3. *$\mathscr{D}_A$ is monoidally Morita equivalent to the Weyl algebra $\mathscr{D}_{k[t]}$.*

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These algebras provide the first examples of **cocommutative**, **conilpotent** Hopf algebroids in characteristic zero which are not of the form $\mathscr{U}_A(L)$, and whose conilpotent filtration is not a coalgebra filtration.

Non-examples: Nodal Curves

- $I := \mathfrak{M}_1 \cdots \mathfrak{M}_r \subseteq k[t]$, where $\mathfrak{M}_1, \ldots, \mathfrak{M}_r$ are pairwise distinct maximal ideals
- $A := k + I$ subalgebra of $k[t]$

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Theorem (M. 2026)

1. *$\mathscr{D}_A$ is not $R(A)$-locally projective.*
2. *$\mathscr{D}_A$ does not admit a Hopf algebroid structure.*

Rings of differential operators: the non-commutative case (work in progress)

Motivation

- There exist several non-equivalent definitions of $\mathscr{D}_A$ in the non-commutative case.

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  - admits a Hopf algebroid structure when A is **nice**.

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  - is intrinsic to A,
  - reduces to Grothendieck's definition when A is commutative, and
  - admits a Hopf algebroid structure when A is **nice**.
- The free algebra is the first step towards our goal.

Differential operators

We assume that A is not necessarily commutative.

Definition (Lunts–Rosenberg 1997)

The **ring of k-linear differential operators** on A is the filtered subalgebra of $\mathrm{End}_k(A)$

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where

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$$\binom{a}{i} \in \mathrm{Der}_k(A), \quad \text{where} \quad \left[\binom{a}{i}, \lambda_{x_j}\right] = \delta_{i,j}\,\lambda_a \text{ and } \binom{a}{i}(1) = 0.$$

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Given a sequence of pairs $\big((a_1, i_1), \dots, (a_n, i_n)\big) \in (A \times \{1, \dots, r\})^n$, $n \geqslant 2$, we define

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For $n = 0$, we define

$$() = \mathrm{id}_A\,.$$

Differential operators on the free algebra

Differential operators on the free algebra

Theorem (Iyer–McCune 2012)

$$\mathscr{D}_A = \mathrm{span}_k\{\lambda_a \mid a \in A\} \cup \left\{ \begin{pmatrix} a_1 & \cdots & a_n \\ i_1 & \cdots & i_n \end{pmatrix} \mid \big((a_1, i_1), \ldots, (a_n, i_n)\big) \in (A \times \{1, \ldots, r\})^n \right\}$$

In particular,

$$\begin{pmatrix} a_1 & \cdots & a_n \\ i_1 & \cdots & i_n \end{pmatrix} \in \mathscr{D}_A^n$$

for all $\big((a_1, i_1), \ldots, (a_n, i_n)\big) \in (A \times \{1, \ldots, r\})^n$.

Bialgebroid structure

Proposition (M. 2026)

$\mathscr{D}_A$ is a bialgebroid over A, with structure maps

$$\begin{aligned}
\eta\colon A^e &\to \mathscr{D}_A, & a \otimes b &\mapsto \lambda_a \rho_b,\\
\Delta\colon \mathscr{D}_A &\to \mathscr{D}_A \otimes_A \mathscr{D}_A, & \begin{pmatrix} a_1 & \cdots & a_n \\ i_1 & \cdots & i_n \end{pmatrix} &\mapsto \sum_{j=0}^{n} \begin{pmatrix} a_1 & \cdots & a_j \\ i_1 & \cdots & i_j \end{pmatrix} \otimes_A \begin{pmatrix} a_{j+1} & \cdots & a_n \\ i_{j+1} & \cdots & i_n \end{pmatrix},\\
\varepsilon\colon \mathscr{D}_A &\to A, & D &\mapsto D(1).
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\eta\colon A^e &\to \mathscr{D}_A, & a\otimes b &\mapsto \lambda_a\rho_b,\\
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\varepsilon\colon \mathscr{D}_A &\to A, & D &\mapsto D(1).
\end{aligned}
$$

Question

Is there a well-known construction which yields this bialgebroid?

Cofree conilpotent Hopf algebras

Let $T^{co}(V)$ be the **cofree conilpotent** coalgebra over a vector space V.

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When is $T^{co}(V)$ a Hopf algebra?

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An **(ungraded)** B_∞**-algebra** structure on V consists of a collection of k-linear maps

$$M_{pq} \colon V^{\otimes p} \otimes V^{\otimes q} \to V,$$

such that $M_{00} = 0$, $M_{10} = \mathrm{id}_V = M_{01}$, $M_{n0} = 0 = M_{0n}$ for $n \geqslant 2$, and satisfying certain relations.

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There exists a bijective correspondence between:

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A B_∞-algebra for which $M_{pq} = 0$ for all $p > 1$ is called a **brace** algebra.

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Lemma (M. 2026)

- $\mathrm{Der}_k(A)$ *is a brace algebra,*
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and with unit map, comultiplication and counit:

$$\begin{aligned} \eta \colon a \otimes b &\mapsto a \otimes 1 \otimes b, \\ \underline{\Delta} \colon a \odot h \odot b &\mapsto (a \odot h_{(1)} \odot 1) \otimes_A (1 \odot h_{(2)} \odot b) \\ \underline{\varepsilon} \colon (a \otimes h \otimes b) &\mapsto a \varepsilon(h) b. \end{aligned}$$

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If H is a Hopf algebra, then $A \odot H \odot A$ admits an antipode

$$a \odot h \odot b \mapsto b \odot S(h) \odot a.$$

Bialgebroid morphism

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Proposition (M. 2026)

We have a surjective bialgebroid morphism

$$\bar{\Phi}\colon A \odot T^{co}(\mathrm{Der}_k(A)) \odot A \twoheadrightarrow \mathscr{D}_A, \quad a \odot h \odot b \mapsto \lambda_a \rho_b \Phi(h).$$

Summary

Summary

Commutative case

- Descent theorem for Hopf algebroids
- Examples: monomial curves
- Counterexamples: nodal curves

Free algebra

- Description of $\mathscr{D}_A$ and its bialgebroid structure
- Brace algebra structure on $\mathrm{Der}_k(A)$
- $A \odot T^{co}(\mathrm{Der}_k(A)) \odot A \twoheadrightarrow \mathscr{D}_A$

Appendix

The cusp $A = k[t^2, t^3]$

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As a k-algebra, $\mathscr{D}_A$ is generated by

$$t^2, \quad t^3 \in \mathscr{D}_A^0, \qquad L_{-2} := t^{-2} D_0(D_0 - 3) \in \mathscr{D}_A^2,$$

$$D_0 := t\partial \in \mathscr{D}_A^1, \qquad L_{-3} := t^{-3} D_0(D_0 - 2)(D_0 - 4) \in \mathscr{D}_A^3.$$

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Counit

$$\varepsilon_A(D_0) = \varepsilon_A(L_{-2}) = \varepsilon_A(L_{-3}) = 0$$

Antipode

$$S(D_0) = -D_0 + 1, \quad S(L_{-2}) = L_{-2}, \quad S(L_{-3}) = -L_{-3}$$

Comultiplication

$$\begin{aligned}
\Delta_A(D_0) &= 1 \otimes_A D_0 + D_0 \otimes_A 1, \\
\Delta_A(L_{-2}) &= 1 \otimes_A L_{-2} + 2D_0 \otimes_A (D_0 - 1)L_{-2} - 2L_1 \otimes_A L_{-3} + L_{-2} \otimes_A 1, \\
\Delta_A(L_{-3}) &= 1 \otimes_A L_{-3} + 3L_{-2} \otimes_A L_{-1} - 3L_{-1} \otimes_A L_{-2} \\
&\qquad + 6D_0 \otimes_A (D_0 - 1)L_{-3} - 6L_1 \otimes_A L_{-2}^2 + L_{-3} \otimes_A 1,
\end{aligned}$$

where $L_1 := tD_0 \in \mathscr{D}_A^1$ and $L_{-1} := t^{-1} D_0(D_0 - 2) \in \mathscr{D}_A^2$.

Relations in $\mathscr{D}_A$ for $A = k\langle x_1, \ldots, x_r \rangle$

We have the following relations in $\mathscr{D}_A$:

$$\rho_a - \lambda_a = \sum_{j=1}^{r} \begin{pmatrix} [x_j, a] \\ j \end{pmatrix}$$

$$\lambda_a \begin{pmatrix} a_1 & \cdots & a_n \\ i_1 & \cdots & i_n \end{pmatrix} = \sum_{j=1}^{r} \begin{pmatrix} [a, x_j] & a_1 & \cdots & a_n \\ j & i_1 & \cdots & i_n \end{pmatrix} + \begin{pmatrix} aa_1 & \cdots & a_n \\ i_1 & \cdots & i_n \end{pmatrix}$$

$$\rho_a \begin{pmatrix} a_1 & \cdots & a_n \\ i_1 & \cdots & i_n \end{pmatrix} = \sum_{j=1}^{r} \begin{pmatrix} a_1 & \cdots & a_n & [a, x_j] \\ i_1 & \cdots & i_n & j \end{pmatrix} + \begin{pmatrix} a_1 & \cdots & a_n a \\ i_1 & \cdots & i_n \end{pmatrix}$$