One-Cross Bundles for Quantum Homogeneous Spaces

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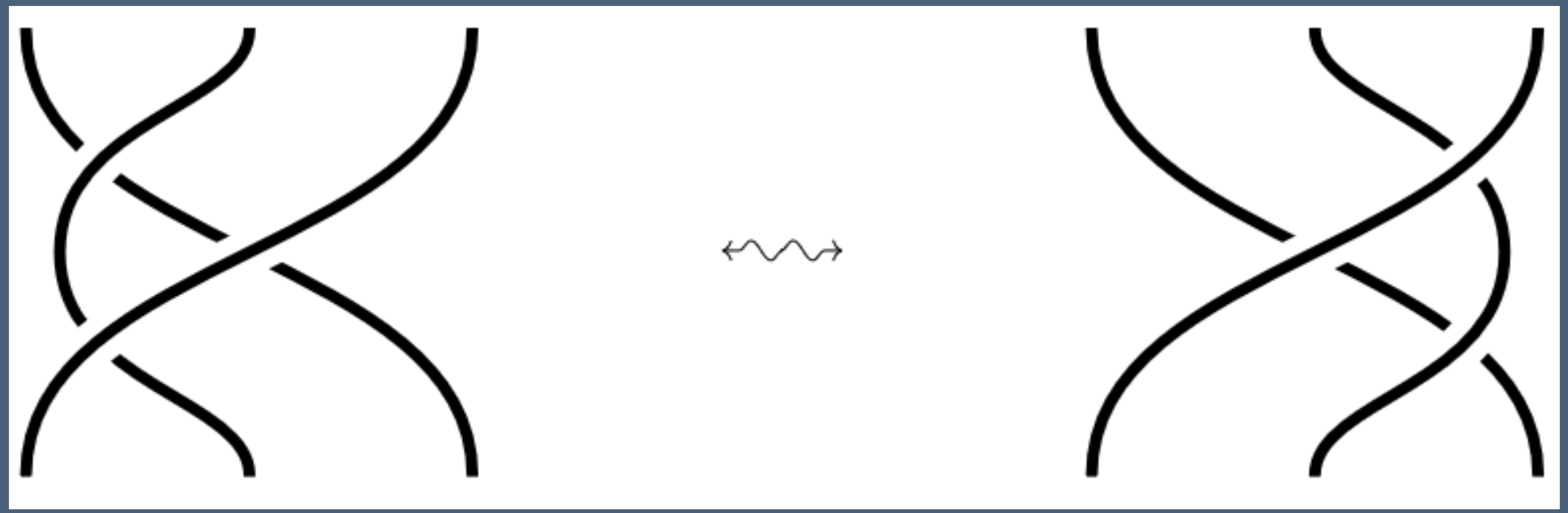
COSUMS 2026 - TU Dresden

The Yang–Baxter Equation

For a linear map $\sigma : V \otimes V \to V \otimes V$ satisfying

$$(\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma)$$



Statistical Mechanics
Exactly Solvable
Temperley–Lieb
Reidemeister Moves
Conformal Field Theory
Bethe Ansatz
R-matrix
Hopf Algebra
Quantum Inverse Scattering
Yangian
Monodromy
Quantum Group
Quantum Double
Hecke Algebra
Knot Invariants
Quasi-triangular
Six-Vertex Model
Jones Polynomial
Ribbon Category
Yang–Baxter Equation
Modular Tensor Category
Algebraic Bethe Ansatz
Vertex Models
Topological QC
Transfer Matrix
Integrability
Boltzmann Weights
S-matrix
Spin Chains
Heisenberg Model
Drinfeld
Ice-Type Model
Spectral Parameter
Anyons
Lax Pair
Crossing Symmetry
Braid Group
Factorized Scattering
Unitarity
Affine Lie Algebra

Hopf Algebras and Quantum Homogeneous Spaces

Quantum homogeneous spaces

- Let A and U be a dual pair of Hopf $*$-algebras.
- Let $L \subset U$ be a Hopf $*$-subalgebra of U.
- Denote the space of invariants by

$$B := {}^{L}A = \left\{ a \in A \mid a_{(1)}\langle a_{(2)}, l\rangle = \epsilon(l)a, \text{ for all } l \in L \right\}.$$

- If $A \otimes_B - : {}_B\mathbf{Mod} \to \mathbf{Vect}_{\mathbb{C}}$ is a faithfully flat functor then we say that B is a quantum homogeneous space.

Example (Hopf fibration from a Hopf algebra point of view): Let

$$A = \mathcal{O}(SU_2), U = U(\mathfrak{sl}_2) \text{ and } L = \mathbb{C}\langle H\rangle.$$

Then $B = \mathcal{O}(S^2)$, the algebra of polynomial functions of the classical 2-sphere.

One-Cross Bundles for QH-Spaces

Definition: Let A and U be a dual pair of Hopf $*$-algebras, and let $L \subseteq U$ be a Hopf $*$-subalgebra such that the space of invariants $B := {}^{L}A$ is a quantum homogeneous space.

A *one-cross bundle* for B is an element $X \in U$, such that

- $\langle X, B \rangle \neq 0$,
- $\Delta(X) = X \otimes K_1 + K_2 \otimes X$, where $K_1, K_2 \in L$ are grouplike elements,
- $T := X \triangleleft_{\mathrm{ad}} L$ is finite-dimensional and simple as a right L-module.

One-Cross Bundles for QH-Spaces

The definition separates into **two** possible cases:

1. ($\mathbb{R}$-case) $X^* \in X \triangleleft_{\mathrm{ad}} L$,

2. ($\mathbb{C}$-case) $X^* \notin X \triangleleft_{\mathrm{ad}} L$.

Here we focus on the **complex case**, but for the real case things progress analogously.

Example: Let $A = \mathcal{O}(SU_2), U = U(\mathfrak{sl}_2)$, $L = \mathbb{C}\langle H \rangle$.

Then $B = \mathcal{O}(S^2)$, and putting $X = E$ gives us a one-cross bundle.

So what can this structure do for quantum homogeneous spaces?

We claim:
From this structure, we can build a noncommutative geometry!

First-Order Differential Calculi

Definition: A ***first-order differential calculus*** over an algebra B is a pair (Ω^1, d), where Ω^1 is a B-bimodule and $\mathrm{d} : B \to \Omega^1$ is a derivation, that is,

$$\mathrm{d}(bc) = (\mathrm{d}b)c + b\mathrm{d}c, \qquad \text{for all } b, c \in B,$$

and we have a surjective multiplication map

$$B \otimes \mathrm{d}B \to \Omega^1(B)\,.$$

Lemma: *For a one-cross bundle* $(A, L \subseteq U, X)$, *the direct sum*

$$T := (X^* \triangleleft_{\mathrm{ad}} L) \oplus (X \triangleleft_{\mathrm{ad}} L)$$

is a quantum tangent space for A, *that is,*

$$\Delta(T) \subseteq (T \oplus \mathbb{C}1_U) \otimes A^\circ.$$

Woronowicz: tangent spaces $\Longleftrightarrow$ first-order differential calculi

Some explicit details on this correspondence:

The space of 1-forms is given by the free A-module

$$\Omega^1(A) := A \otimes \Lambda^1 .$$

The exterior derivative acts explicitly as

$$db = \sum_i (X_i \triangleright b) \otimes e_i, \qquad \text{where } \{X_i\}_i \text{ is a basis of } T, \text{ and } \{e_i\}_i \text{ is a dual basis of } \Lambda^1,$$

We denote the restriction of $\Omega^1(A)$ to B by $\Omega^1(B)$, that is,

$$\Omega^1(B) := \mathrm{span}\left\{ b\mathrm{d}b' \,|\, b, b' \in B \right\}.$$

Example

Example: Let $A = \mathcal{O}(SU_2)$, $U = U(\mathfrak{sl}_2)$, $L = \mathbb{C}\langle H \rangle$. Then $B = \mathcal{O}(S^2)$, which is the algebra of polynomial functions of the 2-sphere.

Let $X = E$, then the tangent space

$$T = \mathrm{span}\{E, F\}$$

is the complexified tangent space of S^2 .

Connections

Definition: Let $\Omega^1(B)$ be a first-order differential calculus for an algebra B. Given a left B-module $\mathscr{F}$, a ***connection*** is a linear map

$$\nabla : \mathscr{F} \to \Omega^1 \otimes_B \mathscr{F}$$

such that

$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla(f), \quad \text{for every } b \in B \text{ and } f \in \mathscr{F}.$$

Definition: A connection ∇ for a B-bimodule $\mathscr{F}$ is said to be a ***bimodule connection*** if

$$\nabla(fb) = \nabla(f)b + \sigma(f \otimes \mathrm{d}b) \text{ for every } b \in B \text{ and } f \in \mathscr{F},$$

for a **necessarily unique** bimodule map

$$\sigma : \mathscr{F} \otimes_B \Omega^1(B) \to \Omega^1(B) \otimes_B \mathscr{F}.$$

Lemma (RÓB, AB '26): *For any one-cross bundle, there exists a covariant bimodule connection*

$$\nabla : \Omega^1(B) \to \Omega^1(B) \otimes_B \Omega^1(B)\,.$$

Theorem: *(RÓB, AC,ADD, RÓB, JR, '26) An explicit formula for the bimodule map* $\boldsymbol{\sigma}$ *is given by*

$$\sigma : \Omega^1(B) \otimes_B \Omega^1(B) \to \Omega^1(B) \otimes_B \Omega^1(B), \qquad \mathrm{d}b \otimes \mathrm{d}c \mapsto \mathrm{d}(b_{(1)} c S(b_{(2)})) \otimes \mathrm{d}b_{(3)}\,.$$

Theorem (AC,ADD, RÓB, JR ’26): *For any one-cross bundle, the bimodule map $\boldsymbol{\sigma}$ of the bimodule connection satisfies the Yang–Baxter equation*

$$(\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma).$$

(Reminder) One-Cross Bundles for QH-Spaces

Definition: Let A and U be a dual pair of Hopf $*$-algebras, and let $L \subseteq U$ be a Hopf $*$-subalgebra such that the space of invariants $B := {}^{L}A$ is a quantum homogeneous space. A ***one-cross bundle*** for B is an element $X \in U$, such that

- $\langle X, B \rangle \neq 0$,
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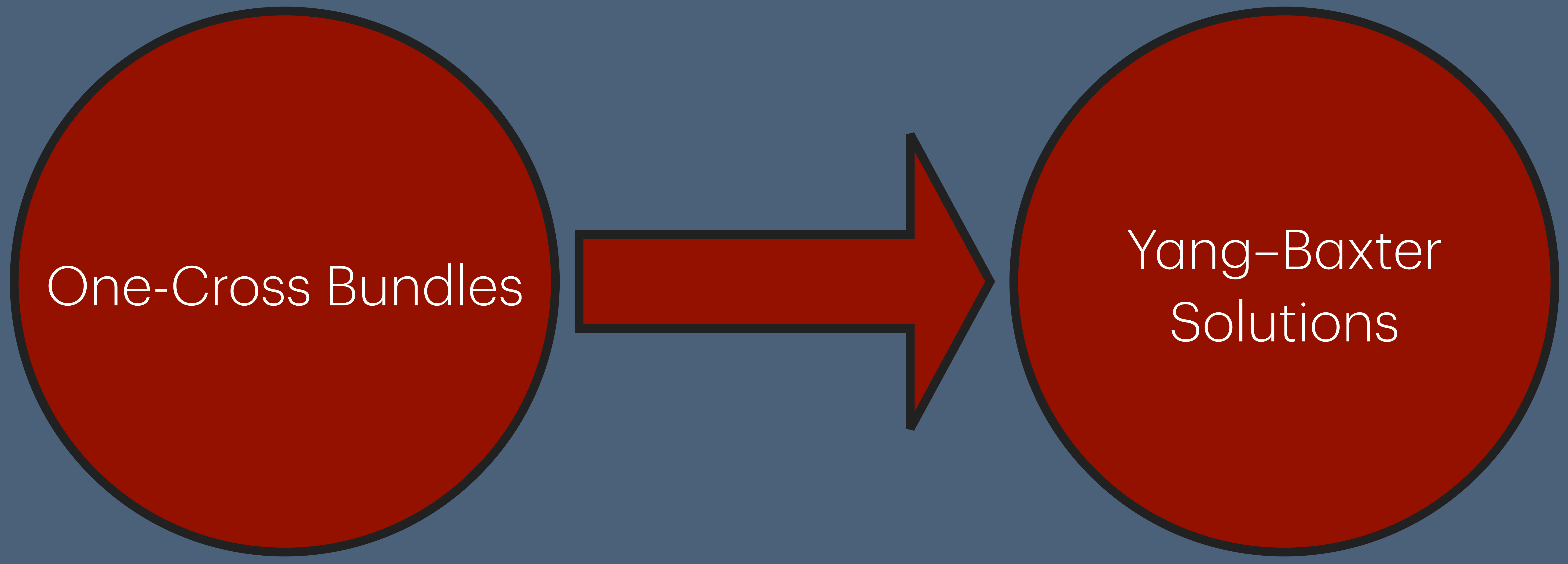

One-Cross Bundles
Yang–Baxter
Solutions

Quantum Levi-Civita Connections

Definition: A ***strong-one-cross bundle*** for B is a weak one-cross bundle $X \in L$ such that L admits a central element Z, such that for any second or third root of unity θ,

$$X \triangleleft Z \neq \theta X\,.$$

Theorem: *The unique covariant connection* $\nabla : \Omega^1 \to \Omega^1 \otimes_B \Omega^1$ *is a Levi-Civita connection with respect to a unique quantum Riemannian metric* g_λ*.*

Examples

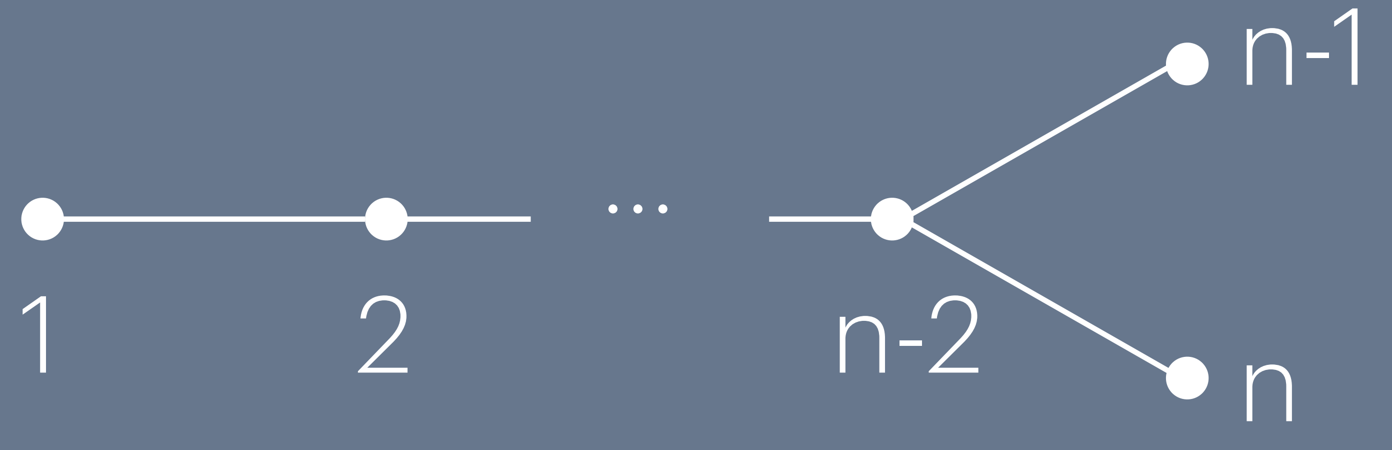
Examples

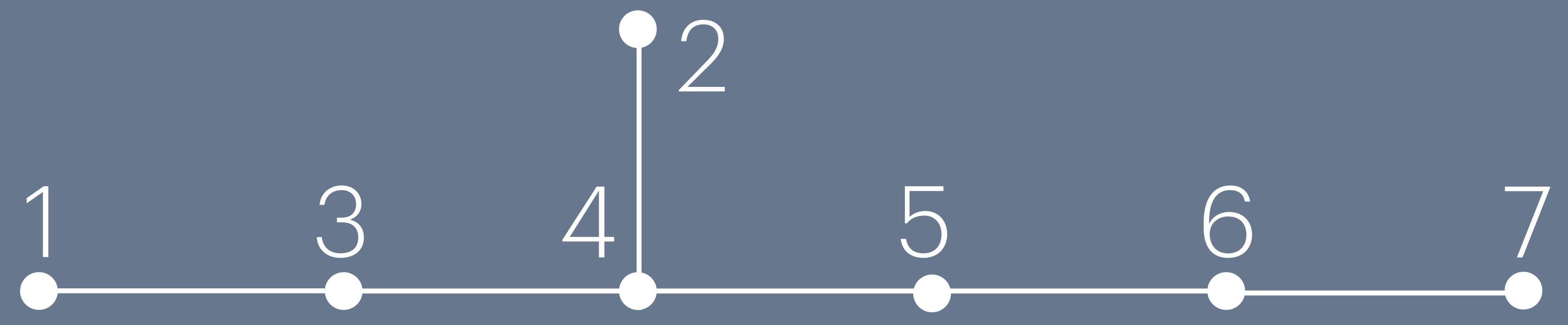
Example: Letting $A = \mathcal{O}_q(SU_2)$, $U = U_q(\mathfrak{sl}_2)$, and $L = \langle K, K^{-1} \rangle$, we get the Podleś sphere. Choosing the element $X = E$ we get a one-cross bundle and recover the Podleś calculus with its well-known bimodule connection.

Example: More generally, if with $A = \mathcal{O}_q(SU_{n+1})$, $U = U_q(\mathfrak{sl}_{n+1})$ and

$$L = \langle E_i, F_i, K_i^{\pm 1} \mid i \neq 1 \rangle,$$

we get quantum projective space $\mathcal{O}_q(\mathbb{C}P^n)$. Setting $X = E_1$, we also get a one-cross bundle with associated differential calculus and covariant bimodule connection.

Root system	Dynkin diagram	Highest weight
A_n		$\alpha_1 + \cdots + \alpha_n$
B_n		$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$
C_n		$2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n$
D_n		$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$

Root system	Dynkin diagram	Highest weight
E_6		$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7		$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$ $+3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4$ $+5\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$
F_4		$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2		$3\alpha_1 + 2\alpha_2$

Example

Example: Let $\mathfrak{g}$ be a complex simple Lie algebra, and G the associated connected simply-connected compact Lie group. If we take $A = \mathcal{O}_q(G)$, $U = U_q(\mathfrak{g})$ and

$$L = U_q(\mathfrak{l}_{\alpha_i}) := \langle E_i, F_i, K_i^{\pm 1} \mid \alpha_i \text{ is not cominiscule} \rangle,$$

then we get the cominiscule quantum flag manifolds $\mathcal{O}_q(G/L_S)$.

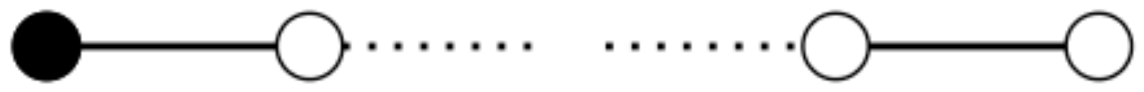
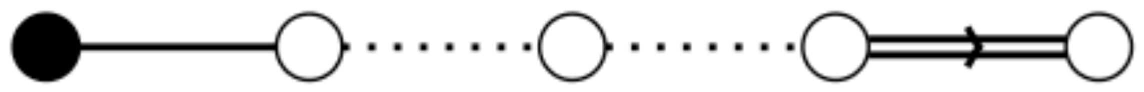
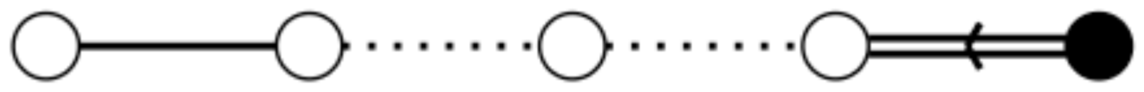
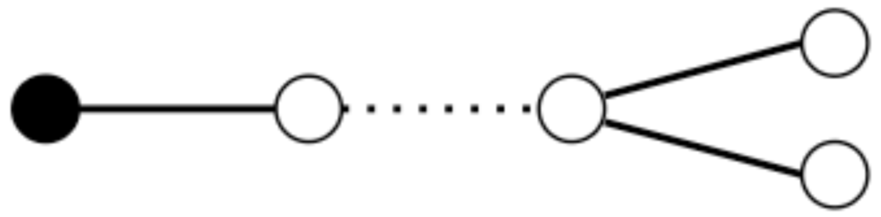
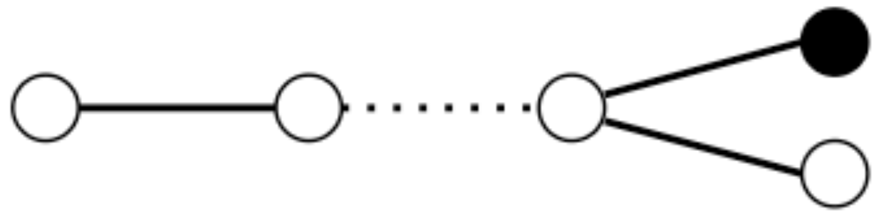
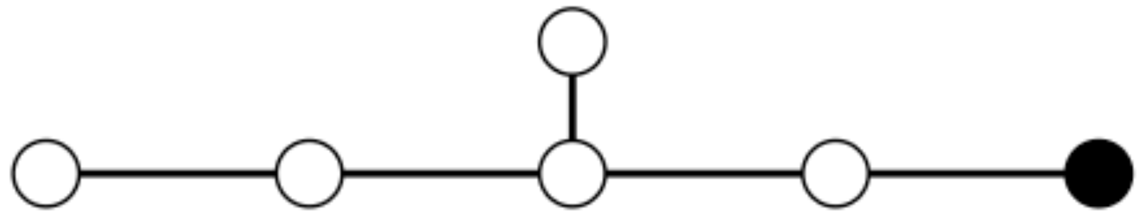
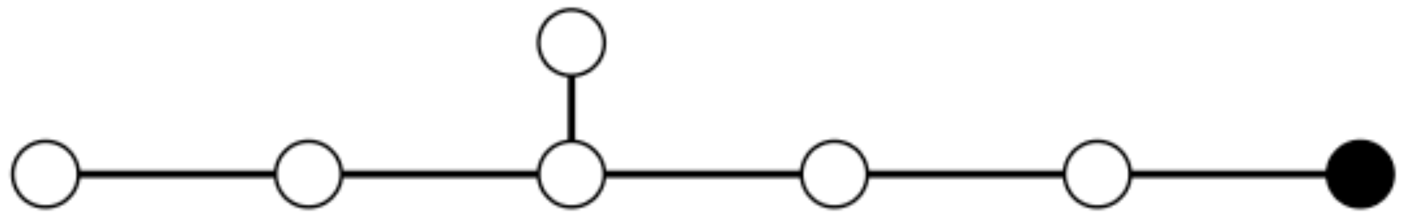
If we set $X = E_x$, for α_x the one-cross root, then we recover the celebrated **Heckenberger–Kolb differential calculi** and their holomorphic structures of their covariant noncommutative associated vector bundles.

Yang–Baxter solutions are rare!

Do the Heckenberger–Kolb braidings appear in the literature ?

$$\underset{\substack{\circlearrowright \\ \sigma}}{\Omega^1_q(G/L_S)} \cong \underset{\substack{\circlearrowright \\ \sigma^{++}}}{\left(\Omega^{(1,0)} \otimes \Omega^{(1,0)}\right)} \oplus \underset{\substack{\circlearrowright \\ \sigma^{+-}}}{\left(\Omega^{(1,0)} \otimes \Omega^{(0,1)} \oplus \Omega^{(0,1)} \otimes \Omega^{(1,0)}\right)} \oplus \underset{\substack{\circlearrowright \\ \sigma^{--}}}{\left(\Omega^{(0,1)} \otimes \Omega^{(0,1)}\right)}$$

Table 1. Does σ coincide with the (rescaled) coquasitriangular braiding of $\mathcal{O}_q(G)$?

Series	Dynkin Diagram	$\mathcal{O}_q(G/L_S)$	$\sigma^{\otimes(2,0)}$	$\sigma^{\otimes(0,2)}$
A_n		$\mathcal{O}_q(\mathbb{CP}^n)$	q	q^{-1}
A_n		$\mathcal{O}_q(\mathrm{Gr}_{x,n+1})$	?	?
B_n		$\mathcal{O}_q(\mathbf{Q}_{2n+1})$	?	?
C_n		$\mathcal{O}_q(\mathbf{L}_n)$	q	q^{-1}
D_n		$\mathcal{O}_q(\mathbf{Q}_{2n})$	q	q^{-1}
D_n		$\mathcal{O}_q(\mathbf{S}_n)$	q	q^{-1}
E_6		$\mathcal{O}_q(\mathbb{OP}^2)$	?	?
E_7		$\mathcal{O}_q(\mathbf{F})$	?	?

Theorem: *It holds that*

$$(\sigma^{+-})^2 = \mathrm{id} \qquad\qquad (\sigma^{-+})^2 = \mathrm{id}.$$

That is, the "mixed braiding" is involutive.

Higher Order Forms

Differential graded algebras and differential calculi

Definition: A *differential graded algebra* over an algebra B is an $\mathbb{N}_0$-graded algebra

$$\Omega^\bullet \cong \bigoplus_{k \in \mathbb{N}_0} \Omega^k,$$

where $\Omega^0 = B$, together with a degree one homogeneous map $\mathrm{d} : \Omega^\bullet \to \Omega^\bullet$ such that

- $\mathrm{d}^2 = 0$, and
- $\mathrm{d}(\omega \wedge \nu) = \mathrm{d}\omega \wedge \nu + (-1)^{|\omega|}\omega \wedge \mathrm{d}\nu,$ for all $\omega, \nu \in \Omega^\bullet$.

If $\Omega^\bullet$ is generated as an algebra by those elements of degree 0 and 1, then we say that it is a *differential calculus* over B.

Extending first-order calculi

QUESTION: How do we extend a first-order differential calculus to a differential calculus?

Example: Let M be a smooth manifold.

The space of 1-forms $\Omega^1(M)$ is a first-order differential calculus over the algebra smooth functions $C^\infty(M)$.

Taking the exterior algebra of $\Omega^1(M)$, we arrive at the de Rham complex, which is a differential calculus over $C^\infty(M)$.

But in the general case, the naïve idea of taking the exterior algebra of a first-order differential calculus will **not** work.

Quotients of tensor algebras

Let $\Omega^1(B)$ be a first-order differential calculus over an algebra B.

Consider the tensor algebra

$$\mathcal{T}(\Omega^1(B)) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} (\Omega^1(B))^{\otimes_B k}.$$

Can we find a B-sub-bimodule

$$N^{(2)} \subseteq \Omega^1(B) \otimes \Omega^1(B)$$

such that the quotient

$$\mathcal{T}(\Omega^1(B)) / \langle N^{(2)} \rangle$$

has the (necessarily unique) structure of a differential graded algebra with map $\mathrm{d} : \mathcal{T}(\Omega^1(B)) / \langle N^{(2)} \rangle \to \mathcal{T}(\Omega^1(B)) / \langle N^{(2)} \rangle$ extending $\mathrm{d} : B \to \Omega^1$?

Maximal prolongation

The answer is **Yes!**

Moreover, there is a **maximal** way to do this.

There exists a unique sub-bimodule $N^{(2)}$ such that, if $\Omega^\bullet = \mathcal{T}(\Omega^1(B))/\langle N^{(2)}\rangle$, then for any other extension of $\Omega^1(B)$ to a differential graded algebra $(\Gamma^\bullet, \delta)$, we have the following commutative diagram:

$$\begin{tikzcd}
 & & \Omega^2(B) \arrow[r, "\mathrm{d}"] \arrow[d] & \Omega^3(B) \arrow[r, "\mathrm{d}"] \arrow[d] & \cdots \\
B \arrow[r, "\mathrm{d}"] & \Omega^1(B) \arrow[ur, "\mathrm{d}"] \arrow[dr, "\delta"'] & & & \\
 & & \Gamma^2 \arrow[r, "\delta"'] & \Gamma^3 \arrow[r, "\delta"'] & \cdots
\end{tikzcd}$$

We call $\Omega^\bullet(B)$ the **maximal prolongation** of $\Omega^1(B)$.

Maximal prolongation for a manifold

Example: Let M be a smooth manifold and $\Omega^1(M)$ the space of one forms.

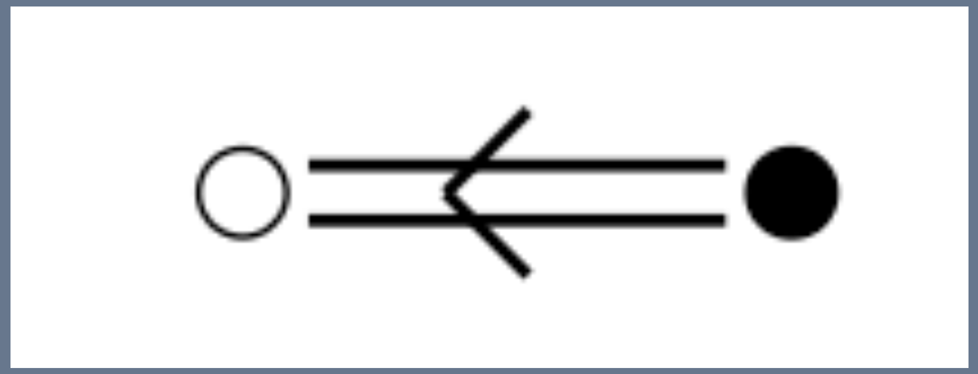
Then the maximal prolongation of $\Omega^1(M)$ corresponds to the usual construction of the de Rham complex as the exterior algebra of $\Omega^1(M)$.

Theorem: *For every cominiscule quantum flag manifold*

$$\mathscr{T}(\Omega^1)/\mathrm{im}(\sigma + \mathrm{id}) = \Omega^\bullet,$$

where $\Omega^\bullet$ *is the maximal prolongation of* Ω^1.

Proposition: *For the* $C_2 / U_q(\mathfrak{sp}_4)$*-cominiscule quantum flag manifold, that is, the quantum Lagrangian Grassmannian* $\mathcal{O}_q(\mathbf{LG}_4)$



there does not exist a Yang–Baxter solution c*, such that*

$$\ker(c - \mathrm{id}) = N^{(2)}.$$

In other words, there is no covariant Nichols algebra description of $\Omega^\bullet$.

What about the non-cominiscule roots?

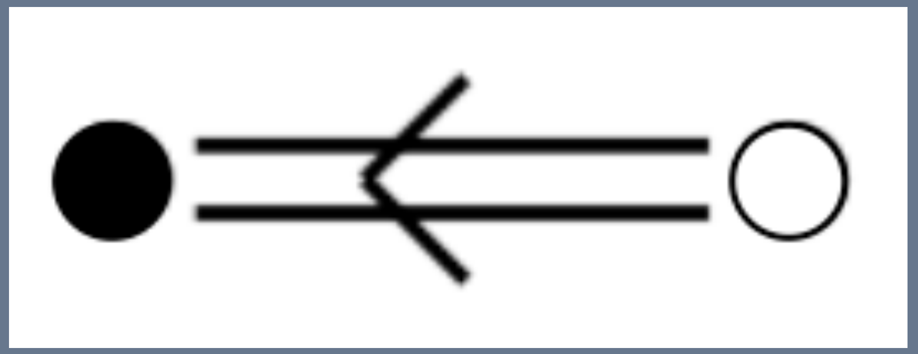
Example

Example: Let $\mathfrak{g}$ be a complex simple Lie algebra, and G the associated connected simply-connected compact Lie group. If we take $A = \mathcal{O}_q(G)$, $U = U_q(\mathfrak{g})$ and

$$L = U_q(\mathfrak{l}_{\alpha_i}) := \langle E_i, F_i, K_j^{\pm 1} \alpha_i \in S, \text{ where } S = \Pi \backslash \{\alpha_i\}) \rangle,$$

for α_x not cominiscule. Setting $X = E_x$, then we get a new one-cross bundle...

Example: Consider the ***quantum isotropic projective space*** $\mathcal{O}_q(\mathrm{I}\mathbb{CP}^4)$



Choosing $X = E_1$ gives us the tangent space $T_q^{(0,1)}$ for which we have the $U_q(\mathfrak{sl}_2)$-module isomorphism

$$T_q^{(0,1)} \cong V_{\varpi_1},$$

and a Yang–Baxter solution that is a scaling of the usual 2-dimensional A-series R-matrix.

The classical tangent space is given by

$$T_q^{(0,1)} \cong V_{\varpi_1} \oplus V_0.$$

The maximal prolongation of the corresponding calculus satisfies

$$\Omega^{(0,k)} = 0, \quad \text{for all } k \geq 2.$$

Example

Example: The B_2/C_2 Weyl group is the dihedral group D_8, the symmetry group of a square. Consider the longest element of D_8

$$s_1s_2s_1s_2 = s_2s_1s_2s_1.$$

The set of Lusztig root vectors corresponding to $s_1s_2s_1s_2$ is

$$\{E_1, \quad E_2, \quad [E_1, E_2]_{q^{-2}}, \quad [E_1, [E_1, E_2]_{q^{-2}}]\}.$$

Proposition (AC, ROB, CP): *Lusztig's root vectors form a tangent space for the full quantum flag manifold of $\mathcal{O}_q(\mathrm{Sp}_4)$. The restriction of the tangent space to $\mathcal{O}_q(\mathrm{I}\mathbb{CP}^4)$ gives*

$$T^{(0,1)} = \{E_1, \quad E_2E_2, \quad [E_1,[E_1,E_2]_{q^{-2}}]\}.$$

The maximal prolongation of the associated calculus has ***classical dimension****.*

Next Step:
Radford–Majid Bosinisation...

Vielen Dank!