Linear coactions of discrete quantum groups on the circle

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Abstract

For a (unital) C^*-algebra $\mathcal{A}$, we construct a C^*-algebraic discrete quantum group (DQG) $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$, coacting on $\mathcal{A}$, which is a quantum generalization of $\mathrm{Aut}(\mathcal{A})$ in the framework of discrete quantum groups, in the sense that any other coaction of a DQG on $\mathcal{A}$ factors through the above coaction of $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$. We prove by an explicit calculation that if any Kac-type C^*-algebraic discrete quantum group $\mathcal{Q}$ has a 'weakly faithful' coaction on $C(S^1)$ which is 'linear' in the sense that it leaves the space spanned by $\{Z, \overline{Z}\}$ invariant, then $\mathcal{Q}$ must be classical, i.e. isomorphic with $C_0(\Gamma)$ for some discrete group Γ. This parallels the well-known result of non-existence of genuine compact quantum group symmetry obtained by the first author and his collaborators ([BG09],[GJ18] and [Gos20])

Construction of the Discrete Quantum Automorphism Group

Fix a unital C^*-algebra $\mathcal{A}$ and consider a category $\widetilde{\mathcal{D}_{\mathcal{A}}}$ whose objects are given by $X = (\alpha, \mathcal{H})$, where $\mathcal{H}$ is a finite dimensional complex Hilbert space and $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\mathcal{H})$ is a unital $*$-homomorphism. Morphisms $\mathrm{Mor}(X, Y)$ for objects $X = (\alpha, \mathcal{H})$ and $Y = (\beta, \mathcal{K})$ are given by linear maps $T : \mathcal{H} \to \mathcal{K}$ such that

$$(1 \otimes T)\alpha(a) = \beta(a)(1 \otimes T) \;\forall a \in \mathcal{A}.$$

Tensor product and direct sum operations are as usual, given by:

$$(\alpha, \mathcal{H}) \oplus (\beta, \mathcal{K}) := (\alpha \oplus \beta, \mathcal{H} \oplus \mathcal{K}),$$
$$(\alpha, \mathcal{H}) \otimes (\beta, \mathcal{K}) := ((\alpha \otimes \mathrm{id}) \circ \beta, \mathcal{H} \otimes \mathcal{K}).$$

The unit object is given by $\mathbb{1} := (\mathrm{id}_{\mathcal{A}}, \mathbb{C})$. It is easy to see that $\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$ for any object $X \in \widetilde{\mathcal{D}_{\mathcal{A}}}$. For any projection $P \in \mathrm{Mor}(X, X)$ we can define the sub-object $X_P = (\alpha_P, P\mathcal{H})$ where $\alpha_P(a) = (\mathrm{id}_{\mathcal{A}} \otimes P)\alpha(a)(\mathrm{id}_{\mathcal{A}} \otimes \iota_{P\mathcal{H}})$ where $\iota_{P\mathcal{H}} : P\mathcal{H} \to \mathcal{H}$ is the inclusion map. Therefore, $\widetilde{\mathcal{D}_{\mathcal{A}}}$ is a C^*-tensor category or unitary tensor category (UTC) which is closed under taking finite direct sums, tensor products and sub-objects.

We call an object $(\alpha, \mathcal{H}_\alpha) \in \widetilde{\mathcal{D}_{\mathcal{A}}}$ to be dualizable if the following conditions hold:
there are $s \in \overline{\mathcal{H}} \otimes \mathcal{H}$, $t \in \mathcal{H} \otimes \overline{\mathcal{H}}$ (where $\overline{\mathcal{H}}$ denotes the complex conjugate Hilbert space of $\mathcal{H}$) and a unital $*$-homomorphism $\hat{\alpha} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\overline{\mathcal{H}})$ satisfying

$$((\hat{\alpha} \otimes \mathrm{id}) \circ \alpha(a))(1 \otimes s) = a \otimes s, \quad ((\alpha \otimes \mathrm{id}) \circ \hat{\alpha}(a))(1 \otimes t) = a \otimes t \quad \forall a \in \mathcal{A},$$
$$(R_s^* \otimes 1_{\overline{\mathcal{H}}})(1_{\overline{\mathcal{H}}} \otimes R_t) = 1_{\overline{\mathcal{H}}}, \quad (R_t^* \otimes 1_{\mathcal{H}})(1_{\mathcal{H}} \otimes R_s) = 1_{\mathcal{H}},$$

where for a vector v in some finite dimensional vector space $\mathcal{K}$ we denote by R_v the map from $\mathbb{C}$ to $\mathcal{K}$ given by $z \mapsto zv$.

We define $\mathcal{D}_{\mathcal{A}}$ to be the subcategory of all dualizable objects in $\widetilde{\mathcal{D}_{\mathcal{A}}}$. Therefore, $\mathcal{D}_{\mathcal{A}}$ being a rigid full subcategory of $\widetilde{\mathcal{D}_{\mathcal{A}}}$ is also closed under tensor product, finite direct sum and sub-objects. We have a canonical fiber functor F from the UTC $\mathcal{D}_{\mathcal{A}}$ to Hilb (the UTC of finite dimensional Hilbert spaces) given by $F(\alpha, \mathcal{H}) = \mathcal{H}$ and $F(T) = T$ for any morphism T. By Tannaka-Krein duality theorem this will give us a CQG or, equivalently the dual DQG.

Definition: Discrete quantum group of automorphism of $\mathcal{A}$

We call the above discrete quantum group as the universal DQG of quantum automorphism for $\mathcal{A}$, denoted by $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$.

Definition: Weakly faithful coaction

Let $\mathcal{S}$ be a C^*-algebraic DQG given by the C^*-algebraic closure of the direct sum of full matrix algebras $\bigoplus_{i \in I} \mathcal{B}(\mathcal{H}_i)$ with a coaction δ on a unital C^*-algebra $\mathcal{A}$. We call the coaction δ to be 'weakly faithful' if the von Neumann algebra generated by the set $\{(\omega \otimes \mathrm{id})\delta(a) : a \in \mathcal{A}, \omega \in \mathcal{A}^*\}$ inside $\mathcal{B}(\oplus_{i \in I} \mathcal{H}_i)$, i.e. the double commutant of the set $\{(\omega \otimes \mathrm{id})\delta(a) : \omega \in \mathcal{A}^*, a \in \mathcal{A}\}$ in $\mathcal{B}(\mathcal{H})$, where $\mathcal{A}^*$ denotes the space of bounded linear functionals on $\mathcal{A}$, is $\mathcal{S}''$.

Coaction of $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$ on $\mathcal{A}$.

We define $\Theta : \mathcal{A} \to \mathcal{M}(\mathcal{A} \otimes \mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})) \cong \prod_{X \in \mathrm{Irr}(\mathcal{D}_{\mathcal{A}})} \mathcal{A} \otimes \mathcal{B}(\mathcal{H}_X)$ by

$$\Theta(a) = \prod_{X \in \mathrm{Irr}(\mathcal{D}_{\mathcal{A}})} \alpha_X(a), \; a \in \mathcal{A}. \tag{1}$$

Θ is a weakly faithful coaction

The map Θ is a weakly faithful coaction of the C^*-algebraic DQG $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$ on $\mathcal{A}$.

Before we state the main theorem establishing the universality of $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$, we state a few results obtained in the process as corollaries. Let $\mathcal{S} = \bigoplus_{i \in I} \mathcal{B}(\mathcal{H}_i)$ be a C^*-algebraic discrete quantum group with a C^*-algebraic coaction $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{M}(\mathcal{A} \otimes_{\min} \mathcal{S})$. Then, the decomposition of $\mathcal{S}$ as a direct sum of matrix algebras $\mathcal{B}(\mathcal{H}_i)$, yields the decomposition of $\delta = \bigoplus_{i \in I} \delta_i$ where $\delta_i = (\mathrm{id} \otimes P_i)\delta$ is the projection onto the ith component of $\mathcal{S}$. Furthermore, we have the following

Lemma

The tuple $(\delta_i, \mathcal{H}_i)$, where δ_i is the ith direct summand of the coaction δ of $\mathcal{S}$ on $\mathcal{A}$ as defined above, is an object in the category $\mathcal{D}_{\mathcal{A}}$.

Corollary

For any C^*-algebraic coaction $\delta : \mathcal{A} \to \mathcal{M}(\mathcal{A} \otimes_{\min} \mathcal{S})$, we have the span density condition, i.e. the linear span of $\delta(\mathcal{A})(\mathrm{id}_{\mathcal{A}} \otimes_{\min} \mathcal{S})$ is norm dense in $\mathcal{A} \otimes_{\min} \mathcal{S}$.

Finally, we have the following theorem establishing the universality of $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$ among all such DQG $\mathcal{S}$ coacting on $\mathcal{A}$.

Theorem: Universal property of $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$

(i) For any C^*-algebraic DQG $\mathcal{S}$ with a coaction $\gamma : \mathcal{A} \to \mathcal{M}(\mathcal{A} \otimes_{\min} \mathcal{S})$ there is a unique non degenerate $*$-homomorphism $\hat{\gamma}$ from $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$ to $\mathcal{S}$ such that $(\mathrm{id} \otimes \hat{\gamma}) \circ \Theta = \gamma$. Moreover, $\hat{\gamma}$ is a DQG morphism.
(ii) If, furthermore, γ is weakly faithful then $\hat{\gamma}$ is surjective.
(iii) $(\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A}), \Theta)$ is the (unique upto isomorphism) universal object in the category $\mathcal{C}_{\mathcal{A}}$ with objects $(\mathcal{G}, \beta)$ where $\mathcal{G}$ is a C^*-algebraic DQG and $\beta : \mathcal{A} \to \mathcal{M}(\mathcal{A} \otimes_{\min} \mathcal{G})$ a C^*-algebraic coaction and morphisms from $(\mathcal{G}_1, \beta_1)$ to $(\mathcal{G}_2, \beta_2)$ being DQG morphisms $\eta : \mathcal{G}_1 \to \mathcal{G}_2$ such that $(\mathrm{id} \otimes \eta) \circ \beta_1 = \beta_2$.

Kac-type quantum automorphism group of $\mathcal{A}$

We concentrate on a quantum subgroup of $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$, namely, the Kac-type quantum automorphism group of $\mathcal{A}$. In order to obtain this subgroup, we consider a full subcategory $\mathcal{D}_{\mathcal{A}}^{\mathrm{Kac}}$ of $\mathcal{D}_{\mathcal{A}}$ by restricting to the objects $(\alpha, \mathcal{H})$ for which one can choose $s = \sum_{i=1}^n \overline{e_i} \otimes e_i$, $t = \sum_{i=1}^n e_i \otimes \overline{e_i}$ for some orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathcal{H}$, with $\{\overline{e_i}\}$ being the corresponding orthonormal basis in $\overline{\mathcal{H}}$. Restricting the fiber functor to the subcategory $\mathcal{D}_{\mathcal{A}}^{\mathrm{Kac}}$, we get $\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac}}(\mathcal{A})$. Moreover, as in the case of $\mathcal{Q}_{\mathrm{Aut}}(\mathcal{A})$ we can define $\Theta^{\mathrm{Kac}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac}}(\mathcal{A})$, which can also be proven to be weakly faithful. The universality of $\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac}}(\mathcal{A})$ follows from the following theorem.

Theorem: Universal Property of $\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac}}(\mathcal{A})$

(i) For any Kac type C^*-algebraic DQG $\mathcal{S}$ with a coaction $\gamma : \mathcal{A} \to \mathcal{M}(\mathcal{A} \otimes_{\min} \mathcal{S})$ there is a unique nondegenerate $*$-homomorphism $\hat{\gamma}$ from $\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{kac}}(\mathcal{A})$ to $\mathcal{S}$ such that $(\mathrm{id} \otimes \hat{\gamma}) \circ \Theta = \gamma$.
(ii) If, furthermore, γ is weakly faithful then $\hat{\gamma}$ is surjective.

Linear and Kac-type DQG coaction on $C(S_1)$

We now consider our $C(S^1)$ as our unital C^*-algebra $\mathcal{A}$. Our aim is to prove that there is no genuine Kac-type C^*-algebraic DQG with a weakly faithful, linear coaction on $C(S^1)$. Let us denote by Z the canonical generator of $C(S^1)$, i.e. $Z(t) = t$ for all $t \in S^1$. $\overline{Z}$ denotes the complex conjugate of Z.

Definition: Linear coaction

A co-action $\beta : C(S^1) \to \mathcal{M}(C(S^1) \otimes \mathcal{Q})$ for some DQG $\mathcal{Q}$ is called linear if $\beta(1) = 1 \otimes 1_{\mathcal{M}(\mathcal{Q})}$ and the linear span $\mathcal{L}$ (say) of $\{Z, \overline{Z}\}$ is invariant under the action of β, in the sense that it is mapped into $\mathcal{L} \otimes \mathcal{Q}$.

$\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac,lin}}(C(S^1))$

Let $\mathcal{D}_{C(S^1)}^{\mathrm{Kac,lin}}$ be the full subcategory of $\mathcal{D}_{C(S^1)}^{\mathrm{Kac}}$ consisting of those objects $(\alpha, \mathcal{H})$ for which there is a choice of $\hat{\alpha}$ such that both α and $\hat{\alpha}$ maps the span $\mathcal{L}$ of $\{Z, \overline{Z}\}$ into $\mathcal{L} \otimes \mathcal{B}(\mathcal{H})$ and $\mathcal{L} \otimes \mathcal{B}(\overline{\mathcal{H}})$ respectively. $\mathcal{D}_{C(S^1)}^{\mathrm{Kac,lin}}$ is a UTC and the restriction of F to this subcategory gives a DQG $\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac,lin}}(C(S^1))$ and there is a weakly faithful, linear coaction, say $\Theta^{\mathrm{Kac,lin}}$ by $\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac,lin}}(C(S^1))$ on $C(S^1)$.

Theorem: Universal Property of $\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac,lin}}(C(S^1))$

(i)For any Kac-type C^*-algebraic DQG $\mathcal{S}$ with a linear coaction $\gamma : C(S^1) \to \mathcal{M}(C(S^1) \otimes_{\min} \mathcal{S})$ there is a unique non degenerate $*$-homomorphism $\hat{\gamma}$ from $\mathcal{Q}_{\mathrm{Aut}}(C(S^1))$ to $\mathcal{S}$ such that $(\mathrm{id} \otimes \hat{\gamma}) \circ \Theta = \gamma$. Moreover, $\hat{\gamma}$ is a DQG morphism.
(ii) If, furthermore, γ is weakly faithful then $\hat{\gamma}$ is surjective.
(iii) $(\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac,lin}}(C(S^1)), \Theta^{\mathrm{Kac,lin}})$ is the (unique upto isomorphism) universal object in the category $\mathcal{C}_{C(S^1)}$ with objects $(\mathcal{G}, \beta)$ where $\mathcal{G}$ is a Kac-type C^*-algebraic DQG and $\beta : C(S^1) \to \mathcal{M}(C(S^1) \otimes_{\min} \mathcal{G})$ a linear C^*-algebraic coaction and morphisms from $(\mathcal{G}_1, \beta_1)$ to $(\mathcal{G}_2, \beta_2)$ being DQG morphisms $\eta : \mathcal{G}_1 \to \mathcal{G}_2$ such that $(\mathrm{id} \otimes \eta) \circ \beta_1 = \beta_2$.

Any Kac-type C^*-algebraic DQG with a 'weakly faithful' linear coaction on $C(S^1)$ is classical

The following theorem is extremely crucial for the derivation of our main result.

Theorem: $\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac,lin}}(C(S^1))$ is classical

$\mathcal{Q}_{\mathrm{Aut}}^{\mathrm{Kac,lin}}(C(S^1))$ is commutative as a C^*-algebra, i.e. isomorphic with $C_0(\Gamma)$ for the discrete group Γ which is the isometry group of the circle with discrete topology, i.e. $\Gamma \cong S_d^1 \rtimes \mathbb{Z}_2$ where S_d^1 is the group S^1 equipped with the discrete topology and the semidirect product is with respect to the canonical $\mathbb{Z}_2$ action where the non trivial element of $\mathbb{Z}_2$ maps $z \mapsto \overline{z}$.

The main result follows as a direct corollary of the previous theorem.

Non-existence of genuine Kac-type and linear DQG symmetry of $C(S^1)$

Let $(\mathcal{S}, \gamma)$ be a Kac type C^*-algebraic DQG coacting on $C(S^1)$ such that the coaction γ is weakly faithful and linear in the sense described previously. Then $\mathcal{S}$ is also commutative as a C^*-algebra and hence is classical, i.e. isomorphic to $C_0(\Gamma)$ for some discrete group Γ.

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