Graph Models for Quantum Weighted Projective Spaces

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Abstract

The C^*-algebras of several noncommutative spaces have been realized as graph C^*-algebras. In this work, we construct explicit graph models for the C^*-algebras of quantum weighted projective spaces with arbitrary weights and in any dimension. Our approach is based on a complete description of all irreducible $*$-representations and the inclusion relations among the kernels of these representations, which determines the topology of the primitive ideal space. The associated graphs allows us to classify the C^*-algebras $C(\mathbb{P}_q(l))$, up to $*$-isomorphism, in terms of the weight vector.

Introduction

Background

- Classically, weighted projective spaces arise by quotienting an odd-dimensional sphere by a circle action with varying integer "weights."
- In noncommutative geometry, the same idea produces **quantum weighted projective spaces**. One takes the Vaksman–Soibelman quantum sphere and keeps the subalgebra fixed by a weighted circle action, producing a C^*-algebra that encodes the "topology" of the quantum space.

The Combinatorial Bridge

- **Graph C^*-algebras** provide a powerful bridge between noncommutative geometry and combinatorics.
- Directed graphs are simple combinatorial objects that make complex operator-algebraic invariants (K-theory, ideal structure, traces) explicitly computable.
- Quantum spaces like spheres, projective spaces, and lens spaces, flag manifolds have already been realized as graph algebras.

Our Contribution

In this work, we extend this combinatorial dictionary to quantum weighted projective spaces by:

- Fully classifying all irreducible $*$-representations and the inclusion relations among their kernels.
- Using these data to explicitly construct the corresponding directed graph.

Graph C^*-algebras

Definition 0.1. Let E be a directed graph. The C^*-algebra of the graph E is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v\}_{v\in E^0}$, and partial isometries $\{s_e\}_{e\in E^1}$ with mutually orthogonal ranges, subject to the relations

1. $s_e^*s_e = p_{r(e)}$,
2. $p_{s(e)}s_es_e^* = s_es_e^*$ for every $e \in E^1$,
3. $p_v = \sum_{e\in s^{-1}(v)} s_es_e^*$ for every $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$.

A graph is called an *amplified graph* if there are either no edges between any two given vertices, or there are infinitely many.

Example 0.2. The following are the graphs of the quantum 5-sphere $C_q(S^5)$ and the quantum complex projective space $C_q(\mathbb{C}P^2)$.

Our construction of the graph models for the C^*-algebras of quantum weighted projective spaces relies on the following theorem in [1].

Theorem 0.3. *Any separable unital type 1 C^*-algebra with finite primitive ideal space is isomorphic to the C^*-algebra of a graph.*

The construction of the underlying graph is also described in [1]. The vertices of the graph are given by the points in the primitive ideal space and the edges are given by the inclusion relations.

Quantum Weighted Projective Spaces

Let $n \geq 1$ and $q \in (0,1)$. Let $\mathcal{O}_q(S^{2n+1})$ be the quantized coordinate algebra of the odd dimensional sphere generated by $2n+1$ generators $z_i, i = 1,\dots,n+1$ and their adjoints z_i^*. For $(l_1,\dots,l_{n+1}) \in \mathbb{N}^{n+1}$, define a $U(1)$-action on $\mathcal{O}_q(S^{2n+1})$ by

$$\alpha_t(z_i) = t^{l_i}z_i,$$

for all $i = 1,\dots,n$ and $t \in \mathbb{T}$. The fixed point algebra

$$\mathcal{O}(S_q^{2n+1})^\alpha = \{a \in \mathcal{O}(S_q^{2n+1}) : \alpha_t(a) = a \quad \forall t \in \mathbb{T}\}$$

is called the coordinate algebra of the *quantum weighted projective space* with weight vector l, denoted by $\mathcal{O}(\mathbb{P}_q(l))$. This $*$-algebra admits a C^* completion which we denote by $C(\mathbb{P}_q(l))$.

Irreducible Representations

The irreducible representations of $C_q(S^{2n+1})$ has already been studied. Being a subalgebra, all the representations of $C(\mathbb{P}_q(l))$ are the irreducible components of these representations obtained by restriction. The splitting of these representations depends on the weight vector.

Theorem 0.4 (K–NS). *Let $\psi_m, m = 1,\dots,n+1$ be the irreducible representations of $C_q(S^{2n+1})$. The restriction of ψ_m decomposes as a direct sum of $\tilde{l_m} = \frac{l_m}{\gcd(l_1,\dots,l_m)}$ many irreducible components for $C(\mathbb{P}_q(l))$. The collection*

$$\{\psi_m^{(j)} : m = 1,\dots,n+1; j = 0,\dots\tilde{l}_m - 1\}$$

is a complete list of irreducible representations of $C(\mathbb{P}_q(l))$.

Here each ψ_m^j act on $l^2(L_j) \subseteq l^2(\mathbb{N}_0^m)$ and $L_1 \sqcup \dots \sqcup L_{\tilde{l_m}} = \mathbb{N}_0^m$.

Inclusion relations among kernels

As we do not have a complete set of relations for our C^*-algebra, we find the inclusion relations among the kernel to be highly non-trivial. Our approach to solving this is by embedding the images of the irreducible representations into a universal C^*-algebra. The following are the important steps in the proof:

- Consider a special commutative C^*-subalgebra $D(L)$ associated with $L \subseteq \mathbb{N}_0^m$ of $B(l^2(\mathbb{N}_0^m))$.
- Define a partial action of a group $P \subseteq \mathbb{Z}^m$ on $D(L)$ and construct the partial crossed product $D(L) \rtimes P$.
- Embed $D(L) \rtimes P$ into $B(l^2(\mathbb{N}_0^m))$ so that it contains the image of the corresponding representation.
- Define a $*$-homomorphism between the images of irreducible representations by using the universal property of the partial crossed product with respect to covariant representations.

This is summarized in the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Im}(\psi_m^j) \arrow[r] \arrow[dd, dashed] & D(L_j) \rtimes P \subseteq B(l^2(\mathbb{N}_0^m)) \arrow[dd] \\
C(P_q(l)) \arrow[ur] \arrow[dr] & & \\
 & \mathrm{Im}(\psi_{m-1}^i) \arrow[r] & B(l^2(K_i)) \subseteq B(l^2(\mathbb{N}_0^{m-1}))
\end{tikzcd}$$

The condition for the inclusion among kernels can also be described just in terms of the weight vector. The kernel of ψ_m^j is contained in the kernel of ψ_{m-1}^i if and only if $\gcd((\tilde{l_k}, \tilde{l}_{k+1})$ divides $i - j$.

Main result

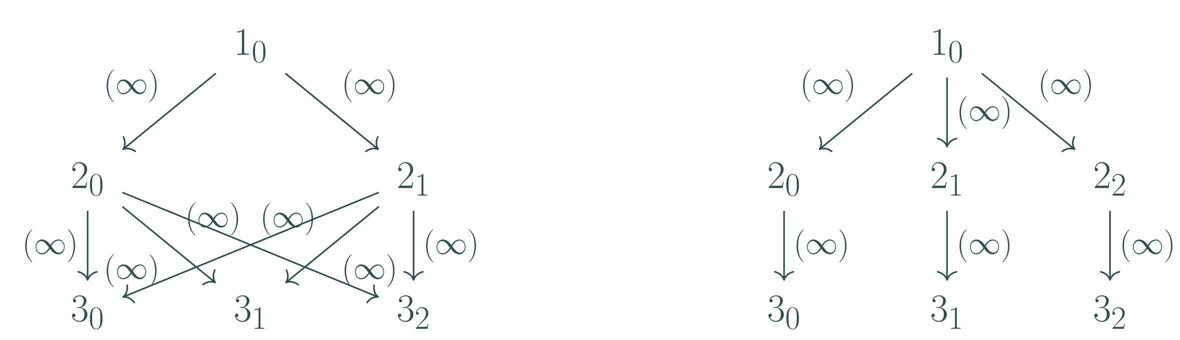
The C^*-algebra of quantum weighted projective spaces for any weight vector l is isomorphic to a C^*-algebra of an amplified graph.

Description of the Graph

- The graph of $C(\mathbb{P}_q(l))$ is a layered amplified graph with $n+1$ layers.
- The i-th layer contains $\frac{l_i}{\gcd(l_1,\dots,l_i)}$ many vertices.
- There is no edges between vertices on the same level and non-adjacent layers.
- There is an edge from the i-th vertex in the k-th level to the j-th vertex in $k+1$-th level iff $\gcd(\tilde{l_k}, \tilde{l}_{k+1})$ divides $i - j$.

Examples

The following are the graphs corresponding to the weight vectors $(1,2,3)$ and $(1,3,3)$, respectively.



Conclusions

- The C^*-algebras of the quantum weighted projective spaces for arbitrary weights and dimensions can be realized as graph C^*-algebras.
- The C^*-algebras corresponding to two weights $l = (l_1,\dots,l_{n+1})$ and $k = (k_1,\dots,k_{n+1})$ are isomorphic if and only if $\tilde{l}_i = \tilde{k}_i, i = 1,\dots,n+1$.
- The C^*-algebras of quantum weighted projective spaces are AF-algebras.
- $K_0(C(\mathbb{P}_q(l))) \cong \mathbb{Z}^{\tilde{l}_1+\dots+\tilde{l}_{n+1}}, \qquad K_1(C(\mathbb{P}_q(l))) \cong 0.$

References

[1] Søren Eilers, Efren Ruiz, and Adam PW Sørensen. Amplified graph c*-algebras. *arXiv preprint arXiv:1110.2758*, 2011.

[2] Aaron Kettner and Akhila Nelliyamkunnath Satheesan. Graph models for quantum weighted projective spaces. In preparation, 2025.

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