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# $C^*$-Algebras and Subproduct Systems

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## Relations & Subproduct Systems

Classically, polynomial relations define varieties. In the quantum setting, homogeneous spaces are encoded by noncommutative coordinate algebras, often arising from quantum group symmetries or coideal subalgebras.

### Goal of the project:

This project investigates how defining relations of noncommutative coordinate algebras can be translated into $C^*$-algebraic structures via subproduct systems, as introduced in [4].

Let $\{x_1, \ldots, x_m\}$ be a set of variables generating $\mathbb{C}\langle x_1, \ldots, x_m\rangle$ and $I$ be an homogenous ideal of relations with degree $l$. We identify the variables with an orthonormal basis $\{e_i\}_{i=1,\ldots,m}$ of an $m$-dimensional Hilbert space $H$ and the $I$ with a subspace $R \subset H^{\otimes l}$.

### Subproduct Systems:

Based on [3, 4], we use the following definitions.

**Definition:** A **subproduct system** is a family of Hilbert spaces $(X_n)_{n\in\mathbb{N}}$ such that

$$X_0 = \mathbb{C} \text{ and } X_{i+j} \subseteq X_i \otimes X_j.$$

Given $H$ and $R$, the construction

$$X_0 := \mathbb{C}, \quad X_n := H^{\otimes n} \text{ for } 1 \leq n < l, \quad X_n := \left(\sum_{i=0}^{n-l} H^{\otimes i} \otimes R^{\perp} \otimes H^{n-l-i}\right)^{\perp} \text{ for } n \geq l$$

gives rise to a subproduct system of finite-dimensional Hilbert spaces.

## Corresponding $C^*$-Algebras

### Fock Spaces:

**Definition:** The **full Fock space** is defined as:

$$\mathcal{F} := \bigoplus_{n\geq 0} H^{\otimes n}.$$

This is the resulting space given no relations.

**Definition:** The **Fock space associated to the subproduct system** $X$ is defined as:

$$\mathcal{F}_X := \bigoplus_{n\geq 0} X_n.$$

As an example of quadratic relations studied in [1], the **symmetric Fock space** for commuting variables $\mathcal{F}_{\text{sym}} = \text{Sym}(H)$ arises from $R_{\text{sym}} := \text{span}\{e_i \otimes e_j - e_j \otimes e_i \mid i < j\} \subset H^{\otimes 2}$.

### Shift Operators:

**Definition:** For orthonormal basis elements $e_i \in H$ and $\eta \in X_n$, we define the creation operator $T_i \in B(\mathcal{F}_X)$:

$$T_i(\eta) := P_{X_{n+1}}(e_i \otimes \eta)$$

where $P_{X_{n+1}}\colon H \otimes X_n \to X_{n+1}$ is an orthogonal projection.

### The Toeplitz Algebra & Extension:

The **Toeplitz algebra** is defined as $\mathcal{T}_X := C^*(T_1, \ldots, T_d)$ and is independent of the choice of a basis. For finite-dimensional Hilbert space subproduct systems, we have the ideal $\mathcal{K}(\mathcal{F}_X) \subset \mathcal{T}_X$ as seen in [5].

**Definition:** The Cuntz–Pimsner algebra $\mathcal{O}_X$ of the subproduct system is defined as the quotient in the Toeplitz extension

$$0 \longrightarrow \mathcal{K}(\mathcal{F}_X) \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Note that in general the Toeplitz algebra $\mathcal{T}_X$ is not $KK$-equivalent to the complex numbers $\mathbb{C}$. It is an open problem (cf. Question (3) in [5]) to determine under which conditions $\mathcal{T}_X$ is $KK$-equivalent to the complex numbers.

## References


[1] F. Arici and G. Yufan. Quadratic subproduct systems, free products, and their c*-algebras. *Integral Equations and Operator Theory*, 98, 04 2026.

[2] S. Bhattacharjee, O. Giselsson, and S. Neshveyev. Cartan subproduct systems, 2025. arXiv:2512.17690.

[3] M. Hartz and O. M. Shalit. Tensor algebras of subproduct systems and noncommutative function theory. *Canad. J. Math.*, 76(5):1587–1608, 2024.

[4] O. M. Shalit and B. Solel. Subproduct systems. *Doc. Math.*, 14:801–868, 2009.

[5] A. Viselter. Cuntz-Pimsner algebras for subproduct systems. *Internat. J. Math.*, 23(8):1250081, 32, 2012.


## K-Theory

### The Six Term Exact Sequence:

The Toeplitz extension leads to the **six term exact sequence** in K-Theory:

$$\begin{tikzcd}
K_0(\mathcal{K}(\mathcal{F}_X)) \arrow[r] & K_0(\mathcal{T}_X) \arrow[r] & K_0(\mathcal{O}_X) \arrow[d, "\delta_0"] \\
K_1(\mathcal{O}_X) \arrow[u, "\delta_1"] & K_1(\mathcal{T}_X) \arrow[l] & K_1(\mathcal{K}(\mathcal{F}_X)) \arrow[l]
\end{tikzcd}$$

For finite-dimensional Hilbert spaces, the extension is well-behaved. In infinite dimensions, however, the ideal $\mathcal{I}_X$ in the extension $\mathcal{O}_X \simeq \mathcal{T}_X/\mathcal{I}_X$ can be larger than $\mathcal{K}(\mathcal{F}_X)$ obstructing the standard K-theoretic computations.

## Example: $q$-Commuting Homogeneous Coordinates

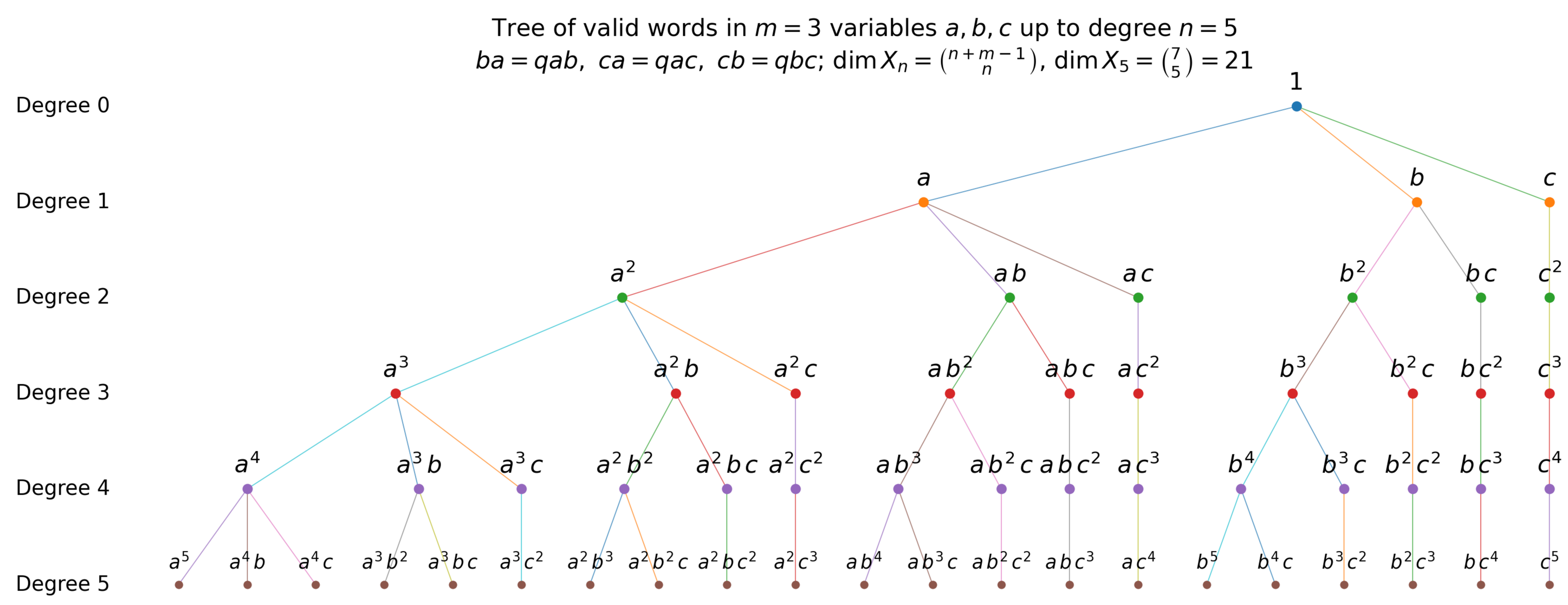
Let $H = \mathbb{C}^m$ be a Hilbert space with orthonormal basis $\{e_1, \ldots, e_m\}$. For $q \in (0, 1]$, let $R$ be the space of quadratic relations (see [1] for the general framework):

$$R = \text{span}\{e_i \otimes e_j - qe_j \otimes e_i \mid i < j\} \subset H^{\otimes 2}.$$

Then

$$X_n = \text{span}\{x_1^{d_1} \ldots x_n^{d_m} \mid d_1 + \ldots + d_m = n\}.$$

The corresponding Fock space is the $q$-symmetric Fock space $\mathcal{F}_X = \text{Sym}_q(H)$.




Choosing the representation $\pi_n\colon S_n \to B(H^{\otimes n})$ of the symmetric group $S_n$ given by the $q$-flips, we obtain the projections

$$p_n = \frac{1}{n!}\sum_{\sigma\in S_n} \pi_n(\sigma) \text{ and } X_n = p_n H^{\otimes n}.$$

The corresponding shift operators in $\mathcal{T}_X$ for $D_m = d_1 + \ldots + d_m$ and $D_{i-1} = d_1 + \ldots + d_{i-1}$ are given by

$$T_i(x_1^{d_1} \ldots x_m^{d_m}) = q^{-D_{i-1}} x_1^{d_1} \ldots x_i^{d_i+1} \ldots x_m^{d_m}, \quad T_i^*(x_1^{d_1} \ldots x_m^{d_m}) = q^{D_m - D_i}\frac{[d_i]_q}{[D_m]_q} x_1^{d_1} \ldots x_i^{d_i-1} \ldots x_m^{d_m}.$$

These have the following relations as the generators of $\mathcal{O}_X$:

$$\sum_{i=1}^{m} S_i S_i^* = 1, \quad S_i S_j = q S_j S_i \ (i < j), \quad S_i^* S_j = q S_j S_i^* \ (i \neq j),$$

$$S_i^* S_i = S_i S_i^* + (1 - q^2)\sum_{j=1}^{i-1} S_j S_j^*$$

which are the relations of the algebra $C(S_q^{2m-1})$ with $S_q^{2m-1} = SU_q(m-1)/SU_q(m)$ as the Vaksman-Soibelman quantum sphere discussed in [2]. This leads to the Toeplitz extension

$$0 \longrightarrow \mathcal{K}(\mathcal{F}_X) \xrightarrow{\varphi} \mathcal{T}_X \xrightarrow{\psi} C(S_q^{2m-1}) \longrightarrow 0$$

and the well-known six term exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{K_0(\varphi)} \mathbb{Z} \xrightarrow{K_0(\psi)} \mathbb{Z} \longrightarrow 0.$$

This Pimsner sequence generalises the Gysin sequence from classical to quantum spheres.

## Outlook & Open Problems

### Explicit Examples:

Find a systematic approach to construct and characterise $C^*$-algebras of subproduct systems arising from arbitrary relations.

### Hilbert $C^*$-modules:

Extend the constructions of subproduct systems and the corresponding projections from Hilbert spaces to Hilbert $C^*$-modules. Determine criteria leading to $\mathcal{K}(\mathcal{F}_X) \subset \mathcal{T}_X$, or determine the ideal $\mathcal{I}_X$ for the relations and Fock spaces to construct the Toeplitz extension and calculate the six term exact sequence.

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