Bounded representations of function algebras on quantum groups and complexification

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Motivation

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Consider the $*$-algebra

$$\mathcal{A} = \mathbb{C}[z, z^{-1}]$$

of Laurent polynomials, with the $*$-structure determined by $z^* = z^{-1}$.

Recall that a *character* of $\mathcal{A}$ is a nonzero algebra homomorphism $\mathcal{A} \to \mathbb{C}$. By a $*$-*character* we mean a character which is also a $*$-homomorphism.

Point evaluation of functions yields canonical identifications

$$\{*\text{-characters } \mathcal{A} \to \mathbb{C}\} \cong S^1$$

and

$$\{\text{characters } \mathcal{A} \to \mathbb{C}\} \cong \mathbb{C}^\times.$$

Note that $\mathbb{C}^\times$ is the *complexification* of the circle group S^1.

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Forgetting the $*$-structure of a (finitely generated) commutative $*$-algebra corresponds to passing to the complexification of the corresponding real algebraic variety.

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In the setting of C^*-algebras, this does not quite work out in the same way.

We have a canonical inclusion $\mathcal{A} = \mathbb{C}[z, z^{-1}] \subset C(S^1) = A$, and by Gelfand duality,

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In particular, every character of $C(S^1)$ is automatically a $*$-character (and continuous).

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In particular, every character of $C(S^1)$ is automatically a $*$-character (and continuous).

To study the complexification of S^1 in the C^*-setting, we need to compare $\mathbb{C}[z, z^{-1}]$ with $C_0(\mathbb{C}^\times)$ instead.

The functions $z, z^{-1} \in \mathbb{C}[z, z^{-1}]$ become *unbounded multipliers* (affiliated elements) of $C_0(\mathbb{C}^\times)$, and $C_0(\mathbb{C}^\times)$ is generated by z, z^{-1} in the sense of Woronowicz.

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Study the standard quantization K_q of a simply connected semisimple compact Lie group K from the perspective of complexification.

We have natural quantum analogous of the algebras appearing in the classical setup, namely

- The coordinate ring $\mathcal{O}(G_q)$ of the standard quantization of G
- The $*$-algebra $\mathcal{O}^{\mathbb{R}}(K_q)$ encoding the compact quantum group K_q and its C^*-completion $C(K_q)$
- The C^*-algebra $C_0(G_q) = C_0(\mathsf{D}(K_q))$ of functions on the Drinfeld double of K_q.

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Study the standard quantization K_q of a simply connected semisimple compact Lie group K from the perspective of complexification.

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Question

To what extent does the interplay between these noncommutative algebras relate to complexification in the classical setting?

Noncommutative algebraic geometry

Noncommutative algebraic geometry

We can think of a finitely generated complex algebra $\mathcal{A}$ as an affine noncommutative scheme/noncommutative complex algebraic variety.

Then

$$\mathrm{Prim}(\mathcal{A}) = \{P \subset \mathcal{A} \mid P \text{ primitive ideal}\}$$

is a natural noncommutative analogue of the maximal ideal space of a commutative algebra.

Recall here that an ideal $P \subset \mathcal{A}$ is called *(left) primitive* iff $P = \ker(\pi)$ where $\pi : \mathcal{A} \to \mathrm{End}(V)$ is an irreducible representation of $\mathcal{A}$ on a complex vector space V, or equivalently, a simple (left) $\mathcal{A}$-module.

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Similarly,

$$\mathrm{Spec}(\mathcal{A}) = \{P \subset \mathcal{A} \mid P \text{ prime ideal}\}$$

is a natural generalisation of the spectrum of a commutative algebra.

Recall here that an ideal $P \subset \mathcal{A}$ is *prime* iff $IJ \subset P$ implies $I \subset P$ or $J \subset P$.

Every primitive ideal is prime.

Some terminology

By a *representation* of an algebra $\mathcal{A}$ we mean an algebra homomorphism $\pi : \mathcal{A} \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the algebra of linear endomorphisms of a complex vector space V. This is the same thing as a (left) $\mathcal{A}$-module.

A representation $\pi : \mathcal{A} \to \mathrm{End}(V)$ is called *nondegenerate* if $\pi(\mathcal{A})V = V$. If $\mathcal{A}$ is unital then π is nondegenerate iff it is unital.

An *intertwiner* between representations $\pi_1 : \mathcal{A} \to \mathrm{End}(V_1)$ and $\pi_2 : \mathcal{A} \to \mathrm{End}(V_2)$ is a linear map $T : V_1 \to V_2$ such that $\pi_2(a)T = T\pi_1(a)$ for all $a \in \mathcal{A}$. We say that π_1, π_2 are *equivalent* if there exists an invertible intertwiner between them.

A representation $\pi : \mathcal{A} \to \mathrm{End}(V)$ on a nonzero vector space V is called *irreducible* if V admits no nontrivial invariant linear subspaces. We write

$$\mathrm{Irr}(\mathcal{A})$$

for the set of equivalence classes of irreducible representations of $\mathcal{A}$.

Some terminology

By a *bounded representation* of an algebra $\mathcal{A}$ we mean an algebra homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$, where $B(\mathcal{H})$ is the algebra of bounded operators on a Hilbert space $\mathcal{H}$.

A bounded representation $\pi : \mathcal{A} \to B(\mathcal{H})$ on a nonzero Hilbert space $\mathcal{H}$ is called *topologically irreducible* if $\mathcal{H}$ admits no nontrivial closed invariant subspaces.

Caveat

Topologically irreducible representations typically fail to be irreducible. In particular, any bounded representation of a finitely generated algebra on an infinite dimensional Hilbert space is reducible for cardinality reasons.

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We shall call an ideal $I \subseteq \mathcal{A}$ *topologically primitive* if it is the kernel of a topologically irreducible bounded representation.

Caveat

There is no reason for a topologically primitive ideal to be primitive.

Some terminology

Now let $\mathcal{A}$ be a $*$-algebra. Then by a $*$-representation of $\mathcal{A}$ we mean a $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Clearly, every $*$-representation is a bounded representation.

Two $*$-representations are called *unitarily equivalent* if there exists a unitary intertwiner between them, that is, a unitary between the underlying Hilbert spaces which satisfies the (algebraic) intertwiner condition.

We write

$$*\text{-}\operatorname{Irr}(\mathcal{A})$$

for the set of all equivalence classes of topologically irreducible $*$-representations of $\mathcal{A}$ modulo unitary equivalence.

Some notation

Some notation

- Fix $q = e^h \in (0,1)$.
- Let $\mathfrak{g}$ be a semisimple complex Lie algebra of rank N with Cartan matrix (a_{ij}).
- Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra.
- Let $\Delta = \Delta^+ \cup \Delta^-$ be the root system with simple roots $\alpha_1, \ldots, \alpha_N \subset \mathfrak{h}^*$.
- Let $(\ ,\ )$ be the bilinear form on $\mathfrak{h}^*$ obtained by rescaling the Killing form such that all short roots α satisfy $(\alpha, \alpha) = 2$.
- Set $d_i = (\alpha_i, \alpha_i)/2$ and $q_i = q^{d_i}$.
- Let $\varpi_1, \ldots, \varpi_N \in \mathfrak{h}^*$ be the fundamental weights.
- Let $\mathbf{P} = \bigoplus_{j=1}^N \mathbb{Z}\varpi_j$ be the weight lattice.
- Let $\mathbf{P}^+ = \bigoplus_{j=1}^N \mathbb{N}\varpi_j$ be the dominant integral weights.
- Let W be the Weyl group of $\mathfrak{g}$.
- Let G be the simply connected complex Lie group with Lie algebra $\mathfrak{g}$

The Drinfeld-Jimbo algebra associated to $\mathfrak{g}$

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The *quantized universal enveloping algebra* $U_q(\mathfrak{g})$ is the algebra with generators E_j, F_j for $1 \leq j \leq N$ and K_λ for $\lambda \in \mathbf{P}$ satisfying

$$K_0 = 1, \qquad K_\lambda K_\mu = K_{\lambda+\mu}$$

$$K_\lambda E_j K_\lambda^{-1} = q^{(\lambda,\alpha_j)} E_j, \qquad K_\lambda F_j K_\lambda^{-1} = q^{-(\lambda,\alpha_j)} F_j$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0 \qquad i \neq j$$

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The Drinfeld-Jimbo algebra associated to $\mathfrak{g}$

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is a Hopf algebra with the comultiplication $\hat{\Delta} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by

$$\hat{\Delta}(K_\mu) = K_\mu \otimes K_\mu,$$
$$\hat{\Delta}(E_i) = E_i \otimes K_i + 1 \otimes E_i$$
$$\hat{\Delta}(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$$

on generators.

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on generators.

It is also a $*$-algebra with the $*$-structure

$$E_i^* = K_i F_i, \qquad F_i^* = E_i K_i^{-1}, \qquad K_\lambda^* = K_\lambda,$$

which turns it into a Hopf $*$-algebra.

With this $*$-structure, we consider $U_q(\mathfrak{g})$ as quantization of the (complex) universal enveloping algebra of the (real) Lie algebra $\mathfrak{k}$, and write $U_q^{\mathbb{R}}(\mathfrak{k})$.

Here $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup $K \subseteq G$.

Representation theory and representative functions

Representation theory and representative functions

For every $\mu \in \mathbf{P}^+$ there exists a unique irreducible representation $V(\mu)$ with a highest weight vector v_μ satisfying

$$K_\lambda v_\mu = q^{(\lambda,\mu)} v_\mu$$

for all $\lambda \in \mathbf{P}$.

Definition

The coordinate ring $\mathcal{O}(G_q) \subset U_q(\mathfrak{g})^*$ of G_q is the Hopf algebra of matrix coefficients of all $V(\mu)$ for $\mu \in \mathbf{P}^+$.

We have a natural quotient homomorphism of Hopf algebras

$$\mathcal{O}(G_q) \to \mathcal{O}(H) \cong \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$$

where $H \subset G$ is the complex torus associated to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

This induces left and right actions of H on $\mathcal{O}(G_q)$.

Representation theory and representative functions

Representation theory and representative functions

The irreducible representations $V(\mu)$ are naturally $*$-representations of $U_q^{\mathbb{R}}(\mathfrak{k})$.

This induces a Hopf $*$-algebra structure on $\mathcal{O}(G_q)$.

We write $\mathcal{O}^{\mathbb{R}}(K_q)$ for $\mathcal{O}(G_q)$ when we want to view it as a Hopf $*$-algebra with this $*$-structure.

Definition

The compact quantum group K_q is given by the (universal) C^*-completion $C(K_q)$ of $\mathcal{O}^{\mathbb{R}}(K_q)$.

$\mathcal{O}^{\mathbb{R}}(K_q)$ is a deformation of the algebra $\mathcal{O}^{\mathbb{R}}(K)$ of representative functions on the compact group K.

We have a canonical $*$-homomorphism

$$\mathcal{O}^{\mathbb{R}}(K_q) \to \mathcal{O}^{\mathbb{R}}(T) \cong \mathbb{C}[z_1^{\pm 1}, \dots, z_N^{\pm 1}]$$

where $T \subset K$ is the maximal torus, and $\mathcal{O}^{\mathbb{R}}(T) = \mathcal{O}(H)$ carries the $*$-structure $z_j^* = z_j^{-1}$.

By construction, the $*$-representations of $\mathcal{O}^{\mathbb{R}}(K_q)$ are in bijective correspondence with the $*$-representations of $C(K_q)$.

Example: the quantum groups $SL_q(2,\mathbb{C})$ and $SU_q(2)$

The coordinate ring $\mathcal{O}(SL_q(2,\mathbb{C}))$ is the unital algebra with generators $\alpha, \beta, \gamma, \delta$ and relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\gamma = \gamma\beta, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma,$$
$$\alpha\delta - q\beta\gamma = 1 = \delta\alpha - q^{-1}\gamma\beta.$$

The $*$-algebra structure on $\mathcal{O}(SL_q(2,\mathbb{C}))$ defining $\mathcal{O}^{\mathbb{R}}(SU_q(2))$ is determined by

$$\alpha^* = \delta, \qquad \beta^* = -q\gamma.$$

This makes the fundamental matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

unitary.

Irreducible $*$-representations

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Firstly, consider the operators on $l^2(\mathbb{N})$ defined by

$$\varphi_q(\alpha)e_k = \sqrt{1-q^{2k}}\,e_{k-1}, \quad \varphi_q(\gamma)e_k = q^k e_k, \quad k \in \mathbb{N},$$

where $e_0, e_1, e_2, \ldots$ is the standard orthonormal basis of $l^2(\mathbb{N})$.

One verifies that this yields a $*$-representation $\varphi_q : \mathcal{O}^{\mathbb{R}}(SU_q(2)) \to B(l^2(\mathbb{N}))$.

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Now fix $1 \leq i \leq N$. Let $U_{q_i}^{\mathbb{R}}(\mathfrak{k}_i) \subseteq U_q^{\mathbb{R}}(\mathfrak{k})$ denote the $*$-subalgebra generated by E_i, F_i, and $K_i^{\pm 1}$. This gives rise to an embedding

$$\tilde{U}_{q_i}^{\mathbb{R}}(\mathfrak{su}(2)) \cong U_{q_i}^{\mathbb{R}}(\mathfrak{k}_i) \to U_q^{\mathbb{R}}(\mathfrak{k}),$$

where $\tilde{U}_{q_i}^{\mathbb{R}}(\mathfrak{su}(2)) \subseteq U_{q_i}^{\mathbb{R}}(\mathfrak{su}(2))$ is the $*$-subalgebra generated by $E, F, K_{\varpi}^{\pm 2}$.

Irreducible $*$-representations

The transpose of this embedding defines a Hopf $*$-algebra homomorphism $\sigma_i : \mathcal{O}^{\mathbb{R}}(K_q) \to \mathcal{O}^{\mathbb{R}}(SU_{q_i}(2))$. In this way we obtain $*$-representation

$$\pi_{s_i} = \varphi_{q_i} \circ \sigma_i : \mathcal{O}^{\mathbb{R}}(K_q) \to B(l^2(\mathbb{N})).$$

Fix $w \in W$, and choose a reduced expression $w = s_{i_1} \cdots s_{i_l}$. Define

$$\pi_w = \pi_{s_{i_1}} \otimes \cdots \otimes \pi_{s_{i_l}} : \mathcal{O}^{\mathbb{R}}(K_q) \to B\big(l^2(\mathbb{N}) \otimes \cdots \otimes l^2(\mathbb{N})\big) \cong B(l^2(\mathbb{N})).$$

This is a $*$-representation which is independent of the choice of the reduced expression up to unitary equivalence.

We also have $*$-characters $\pi_t : \mathcal{O}^{\mathbb{R}}(K_q) \to \mathbb{C}$ induced by the quotient map $\mathcal{O}^{\mathbb{R}}(K_q) \to \mathcal{O}^{\mathbb{R}}(T)$ and point evaluations at $t \in T$.

Theorem (Vaksman-Soibelman, Soibelman, Korogodski-Soibelman)

$$*\text{-}\operatorname{Irr}(\mathcal{O}^{\mathbb{R}}(K_q)) = \{\pi_w \otimes \pi_t \mid w \in W, t \in T\} \cong W \times T.$$

Bounded representations and purely algebraic representations

Let us now forget about the $*$-structure, so pass to $\mathcal{O}(G_q)$. We make two observations:

Firstly, the above construction also gives rise to topologically irreducible *bounded* representations of $\mathcal{O}(G_q)$ by taking $\pi_w \otimes \pi_h$ with $h \in H$, the complex torus.

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Secondly, consider the dense subspace D inside $l^2(\mathbb{N}) \otimes \cdots \otimes l^2(\mathbb{N})$ given by the (algebraic) linear span of the standard orthonormal basis.

It is not hard to check that this subspace is preserved by π_w.

Theorem (Vaksman-Soibelman, Soibelman, Korogodski-Soibelman)

The representations $\pi_w \otimes \pi_t : \mathcal{O}(G_q) \to \mathrm{End}(D \otimes \mathbb{C})$ for $w \in W$ and $t \in T$ are irreducible and mutually inequivalent.

This statement carries over immediately to the representations $\pi_w \otimes \pi_h$ for $w \in W$ and $h \in H$.

However, not every irreducible representation of $\mathcal{O}(G_q)$ is of this form...

The prime spectrum of $\mathcal{O}(G_q)$

By work of Hodges-Levasseur and Joseph we obtain a decomposition

$$\mathrm{Spec}(\mathcal{O}(G_q)) = \coprod_{\mathrm{w} \in W \times W} \mathrm{Spec}_{\mathrm{w}}(\mathcal{O}(G_q)).$$

The set $\mathrm{Prim}(\mathcal{O}(G_q))$ of primitive ideals of $\mathcal{O}(G_q)$ is contained $\mathrm{Spec}(\mathcal{O}(G_q))$, so setting

$$\mathrm{Prim}_{\mathrm{w}}(\mathcal{O}(G_q)) = \mathrm{Prim}(\mathcal{O}(G_q)) \cap \mathrm{Spec}_{\mathrm{w}}(\mathcal{O}(G_q)), \quad \mathrm{w} \in W \times W,$$

one obtains

$$\mathrm{Prim}(\mathcal{O}(G_q)) = \coprod_{\mathrm{w} \in W \times W} \mathrm{Prim}_{\mathrm{w}}(\mathcal{O}(G_q))$$

This can be interpreted using the theory of *H-stratification* of Goodearl-Letzter.

Example: $G_q = SL_q(2, \mathbb{C})$

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We have $W = \mathbb{Z}/2\mathbb{Z} = \{e, w_0\}$. The primitive spectrum is

$$\mathrm{Prim}(\mathcal{O}(SL_q(2,\mathbb{C})) = \coprod_{w_+, w_- \in \{e, w_0\}} \mathrm{Prim}_{(w_+, w_-)}(\mathcal{O}(SL_q(2,\mathbb{C}))$$

with

$$\begin{aligned}
\mathrm{Prim}_{(e,e)}(\mathcal{O}(SL_q(2,\mathbb{C})) &= \{\ker(\pi_h) \mid h \in H = \mathbb{C}^\times\} \\
\mathrm{Prim}_{(w_0,e)}(\mathcal{O}(SL_q(2,\mathbb{C})) &= \{(\beta)\} \\
\mathrm{Prim}_{(e,w_0)}(\mathcal{O}(SL_q(2,\mathbb{C})) &= \{(\gamma)\} \\
\mathrm{Prim}_{(w_0,w_0)}(\mathcal{O}(SL_q(2,\mathbb{C})) &= \{\ker(\pi_{w_0} \otimes \pi_h) \mid h \in H = \mathbb{C}^\times\}
\end{aligned}$$

In addition we have the prime ideals 0 and (β, γ), and we get

$$\begin{aligned}
\mathrm{Spec}_{(e,e)}(\mathcal{O}(SL_q(2,\mathbb{C})) &= \{(\beta,\gamma), \ker(\pi_h) \mid h \in H = \mathbb{C}^\times\} \\
\mathrm{Spec}_{(w_0,e)}(\mathcal{O}(SL_q(2,\mathbb{C})) &= \{(\beta)\} \\
\mathrm{Spec}_{(e,w_0)}(\mathcal{O}(SL_q(2,\mathbb{C})) &= \{(\gamma)\} \\
\mathrm{Spec}_{(w_0,w_0)}(\mathcal{O}(SL_q(2,\mathbb{C})) &= \{0, \ker(\pi_{w_0} \otimes \pi_h) \mid h \in H = \mathbb{C}^\times\}
\end{aligned}$$

Homogeneous versus inhomogeneous primitive ideals

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Let us say that $P \in \mathrm{Prim}(\mathcal{O}(G_q))$ is *homogeneous* if $P \in \mathrm{Prim}_{\mathrm{w}}(\mathcal{O}(G_q))$ for $\mathrm{w} = (w, w)$ for some $w \in W$, and *inhomogeneous* otherwise.

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Theorem (Lee -V.)

Let $\mathcal{O}(G_q)$ be the coordinate ring of the standard deformation of a simply connected semisimple complex Lie group G. Moreover let $P \in \mathrm{Prim}(\mathcal{O}(G_q))$. Then

a) If P is homogeneous then P admits a realisation as kernel of a topologically irreducible bounded representation of $\mathcal{O}(G_q)$.

b) If P is inhomogeneous then there are no bounded representations of $\mathcal{O}(G_q)$ with kernel P.

We note that we can realise a homogeneous primitive ideal P specifically as the kernel of $\pi_w \otimes \pi_h$ for some $w \in W, h \in H$.

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Consider the inhomogeneous primitive ideal $P = (\gamma)$. The quotient

$$\mathcal{A} = \mathcal{O}(SL_q(2, \mathbb{C}))/P$$

is the algebra with generators a, b such that a is invertible and $ab = qba$.

Lemma

The algebra $\mathcal{A}$ does not admit any faithful bounded representations.

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is the algebra with generators a, b such that a is invertible and $ab = qba$.

Lemma

The algebra $\mathcal{A}$ does not admit any faithful bounded representations.

Proof. Assume $a, b \in B(\mathcal{H})$ with a invertible, with bounded inverse, and $ab = qba$. Can assume b quasi-nilpotent, that is, $\sigma(b) = 0$ and not nilpotent.

Let $\xi \in \mathcal{H}$. Since a has a bounded inverse there exists $\alpha > 0$ such that

$$\alpha \|b^m \xi\| \leq \|ab^m \xi\| = q^m \|b^m a \xi\| \leq q^m \|b^m\| \|a\| \|\xi\|.$$

Now take the sup over all ξ with $\|\xi\| = 1$ to get

$$\alpha \|b^m\| \leq q^m \|b^m\| \|a\|.$$

Since $b^m \neq 0$ this implies $\alpha \leq q^m \|a\|$ for all $m > 0$, which is impossible. $\square$

What have we learned so far?

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Classically, every primitive ideal of $\mathcal{O}(G)$ is the kernel of a bounded representation.

In fact, every such ideal is the kernel of an evaluation representation $\mathcal{O}(G) \to \mathbb{C} = B(\mathbb{C})$, given by $f \mapsto f(g)$ for some $g \in G$.

In the quantum case this is no longer true.

Key point

Bounded representations only detect a small part of $\mathrm{Prim}(\mathcal{O}(G_q))$.

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Key point

Bounded representations only detect a small part of $\mathrm{Prim}(\mathcal{O}(G_q))$.

But perhaps the primitive ideals which admit bounded realisations are precisely those which can be obtained via *quantum holomorphic evaluations* using the Podleś-Woronowicz complex quantum groups

$$C_0(G_q) = C_0(\mathsf{D}(K_q))$$

given by the *Drinfeld double* $\mathsf{D}(K_q)$ of K_q?

Complex semisimple quantum groups

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The Pontrjagin dual $\hat{K}_q$ is given by the algebra

$$\mathcal{D}(K_q) = \bigoplus_{\mu \in \mathbf{P}^+} \mathrm{End}(V(\mu)).$$

This is naturally a (multiplier) Hopf $*$-algebra, and it admits an obvious evaluation pairing $\mathcal{D}(K_q) \times \mathcal{O}^{\mathbb{R}}(K_q) \to \mathbb{C}$.

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To the classical group A corresponds a *quotient* $\hat{T}$ of $\hat{K}_q$, obtained from the projection $\mathcal{O}^{\mathbb{R}}(K_q) \to \mathcal{O}^{\mathbb{R}}(T)$. Here T is the classical maximal torus.

Complex semisimple quantum groups

The $*$-algebra $\mathcal{D}(K_q)$ admits a unique C^*-completion, namely,

$$C^*(K_q) = c_0\text{-}\bigoplus_{\mu\in\mathbf{P}^+} B(V(\mu)).$$

Recall also that $\mathcal{O}^{\mathbb{R}}(K_q)$ admits a canonical C^*-completion $C(K_q)$.

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Definition

The algebra of functions on the complex quantum group G_q is the C^*-algebra

$$C_0(G_q) = C(K_q) \otimes C^*(K_q).$$

(Irreducible) $*$-representations of $C_0(G_q)$ correspond to commuting pairs of irreducible $*$-representations of $C(K_q)$ and $C^*(K_q)$.

The irreducible $*$-representations of $C^*(K_q)$ are parametrised by $\mathbf{P}^+$.

Holomorphic polynomials

There is a homomorphism i from $\mathcal{O}(G_q)$ into the unbounded multipliers of $C_0(G_q)$ given by

$$i(f) = f_{(1)} \otimes l^-(f_{(2)}).$$

Here $l^-(f)(g) = (\mathcal{R}^{-1}, f \otimes g)$ denotes l-functionals.

This can be viewed as a quantum analogue of the embedding of holomorphic functions on G into unbounded multipliers of $C_0(G)$.

For every irreducible $*$-representation $\pi = \pi_C \otimes \pi_\lambda$ of $C_0(G_q)$, we obtain an associated bounded representation θ_π by precomposing with i.

We call this the *quantum holomorphic evaluation* associated with π.

Quantum holomorphic evaluations

Theorem (Lee-V.)

Let $\lambda \in \mathbf{P}^+$ and let $\pi = \pi_C \otimes \pi_\lambda$ be an irreducible $*$-representation of $C_0(G_q)$, where $\pi_C : C(K_q) \to B(\mathcal{H}_C)$ is a topologically irreducible $*$-representation. Then $\ker(\theta_\pi)$ is primitive if and only if $\lambda = 0$.

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Consider the subsets

$$\mathrm{Prim}(\theta_\pi) = \{P \in \mathrm{Prim}(\mathcal{O}(G_q)) \mid \ker(\theta_\pi) \subseteq P\} \subseteq \mathrm{Prim}(\mathcal{O}(G_q))$$

To what extent they cover $\mathrm{Prim}(\mathcal{O}(G_q))$ as the irreducible $*$-representation $\pi \in *\text{-}\,\mathrm{Irr}(C_0(G_q))$ varies?

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$$\mathrm{Prim}(\theta_\pi) = \Big\{ P \in \mathrm{Prim}(\mathcal{O}(G_q)) \,\Big|\, \bigcap_{\nu \in \mathbf{P}(\lambda)} \ker(\pi_C \otimes K_{-\nu}) \subseteq P \Big\}.$$

That is, the set $\mathrm{Prim}(\theta_\pi) \subseteq \mathrm{Prim}(\mathcal{O}(G_q))$ is the Zariski closure of the finite subset $\big\{ \ker(\pi_C \otimes K_{-\nu}) \mid \nu \in \mathbf{P}(\lambda) \big\}$.

(In-)decomposabiilty

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Call a bounded representation $\theta : \mathcal{A} \to B(\mathcal{H})$ is *orthogonally indecomposable* if there is no nontrivial orthogonal direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into bounded subrepresentations on closed subspaces $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}$.

Theorem (Lee-V.)

For any irreducible $*$-representation $\pi = \pi_C \otimes \pi_\lambda$ of $C_0(G_q))$, the quantum holomorphic evaluation θ_π is orthogonally indecomposiable.

Recall that every irreducible $*$-representation π_C of $\mathcal{O}^{\mathbb{R}}(K_q)$ has a canonical algebraic model, that is, a corresponding irreducible representation on a dense subspace D of the underlying Hilbert space.

Theorem (Lee-V.)

If $\ker(\pi_C) \in \mathrm{Prim}_{w_0,w_0}(\mathcal{O}(G_q))$ and $\lambda \neq 0$ the representation $\theta_\pi = \pi_C \otimes \pi_\lambda : \mathcal{O}(G_q) \to \mathrm{End}(D \otimes V(\lambda))$ is algebraically decomposable.

THANK YOU!