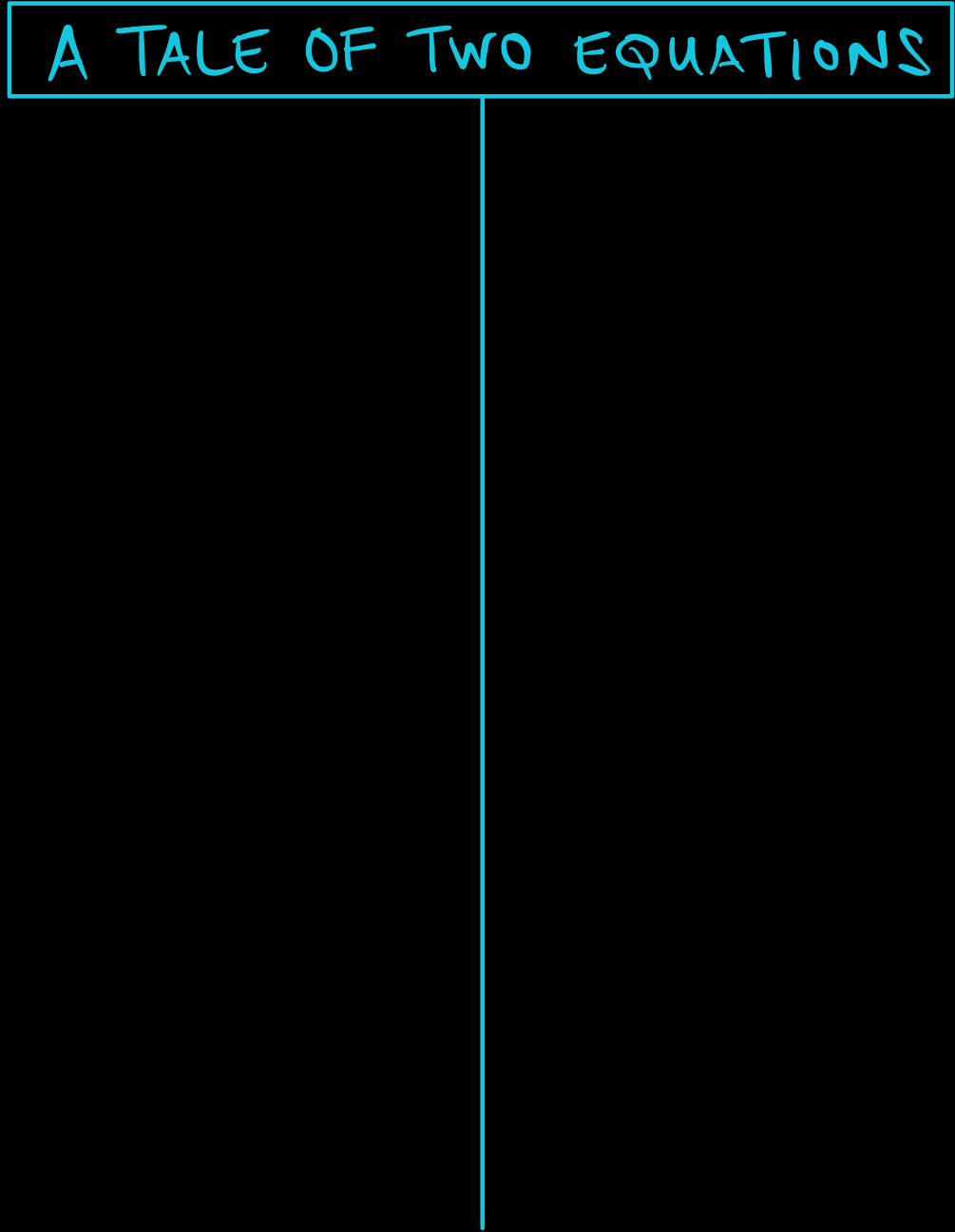


VENTURING TO THE BOUNDARY!

WITH BRAIDED MODULE CATEGORIES

CHELSEA WALTON
RICE UNIVERSITY

A TALE OF TWO EQUATIONS



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graph TD; Title[A TALE OF TWO EQUATIONS] --- Line[ ];
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YANG (1967)
BAXTER (1972)

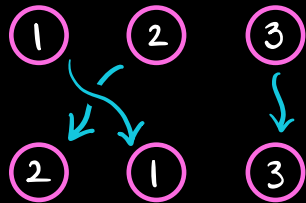
A TALE OF TWO EQUATIONS

↳ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

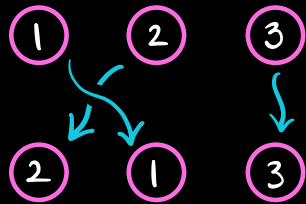
↳ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↳ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



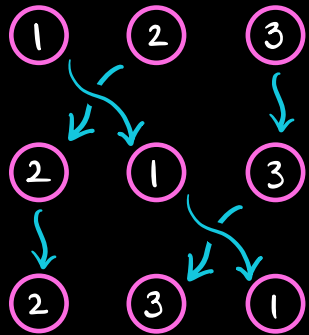
R_{ab} = PARTICLE a PASS PARTICLE b

R_{12}

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↳ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



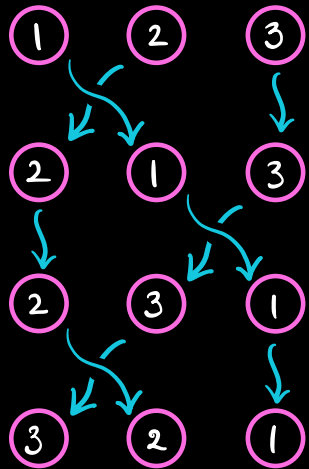
R_{ab} = PARTICLE a PASS PARTICLE b

$R_{12} R_{13}$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↳ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



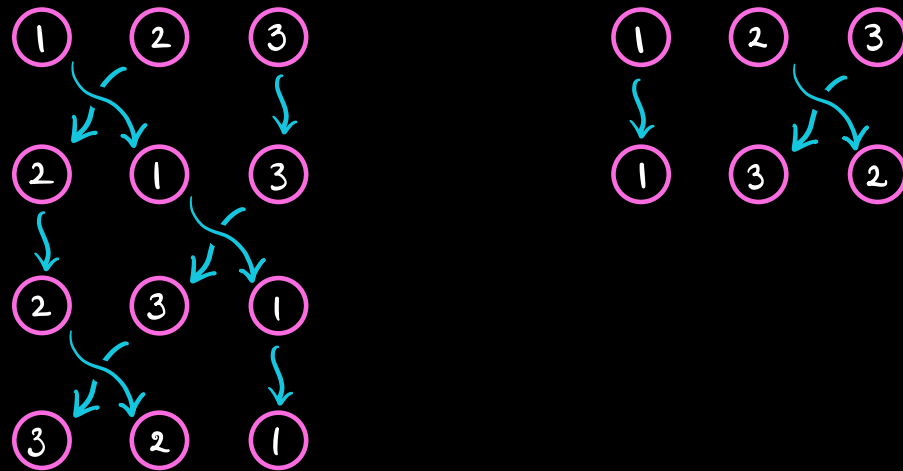
R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23}$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↳ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



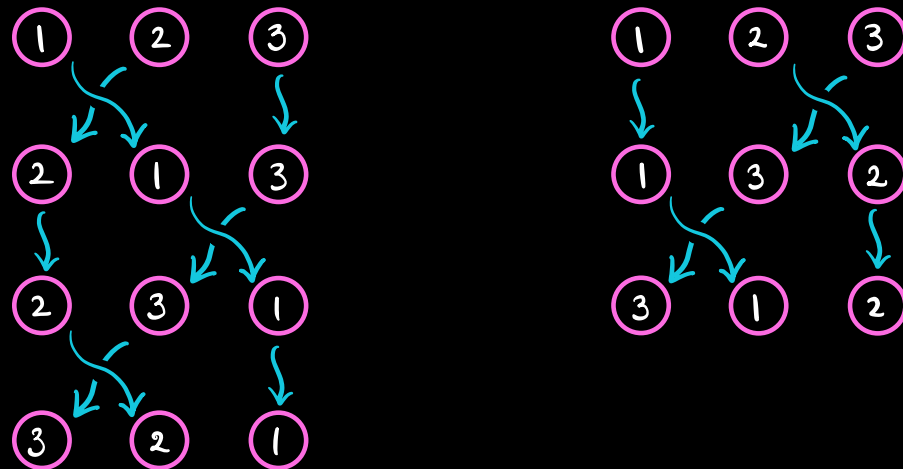
R_{ab} = PARTICLE a PASS PARTICLE b

$R_{12} R_{13} R_{23}$ R_{23}

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↳ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



R_{ab} = PARTICLE a PASS PARTICLE b

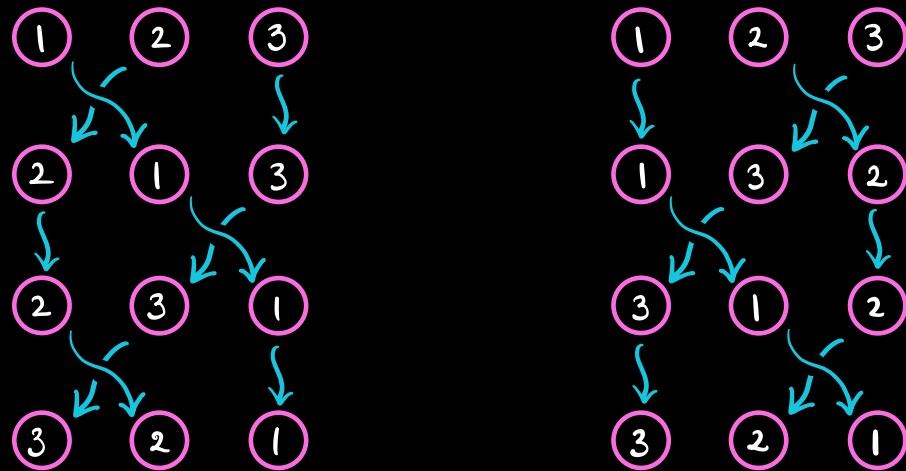
$$R_{12} R_{13} R_{23}$$

$$R_{23} R_{13}$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↳ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



R_{ab} = PARTICLE a PASS PARTICLE b

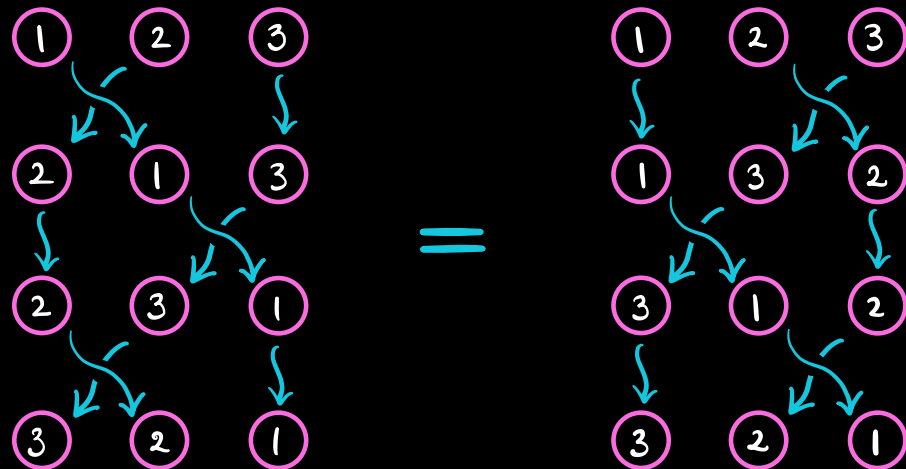
$$R_{12} R_{13} R_{23}$$

$$R_{23} R_{13} R_{12}$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↳ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

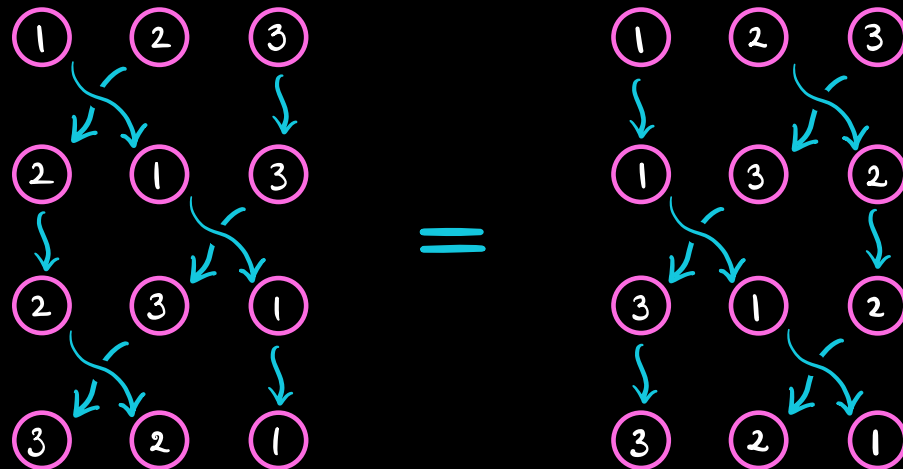
QUANTUM YANG-BAXTER EQ. (QYBE)

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

CHEREDNIK (1984)
SKLYANIN (1988)

↪ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

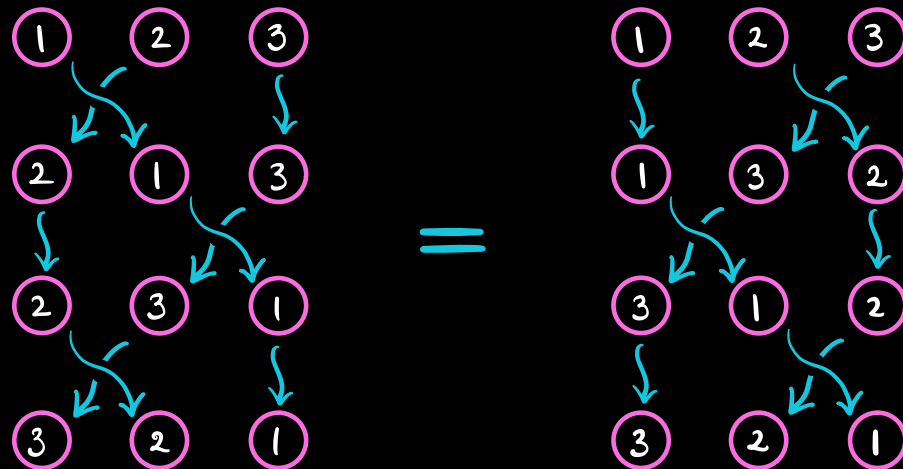
STUDIED HOW PARTICLES ↪
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

CHEREDNIK (1984)
SKLYANIN (1988)

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SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY

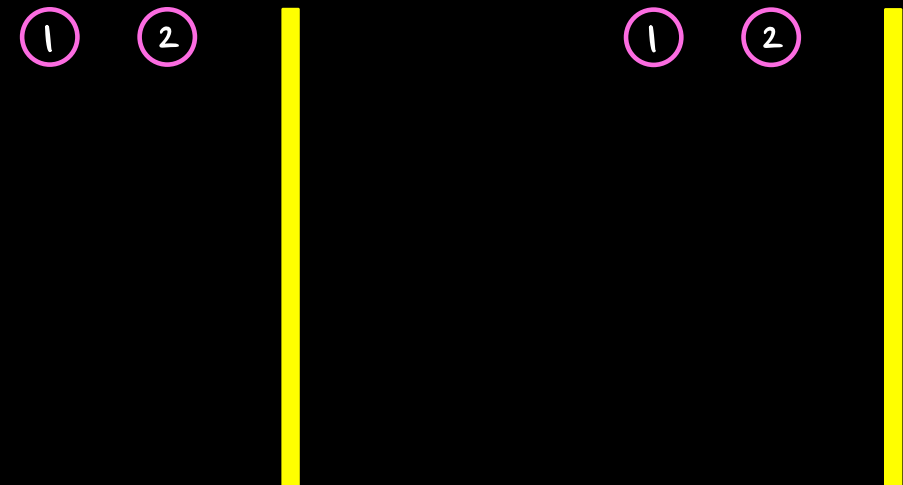


R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

STUDIED HOW PARTICLES ↪
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY



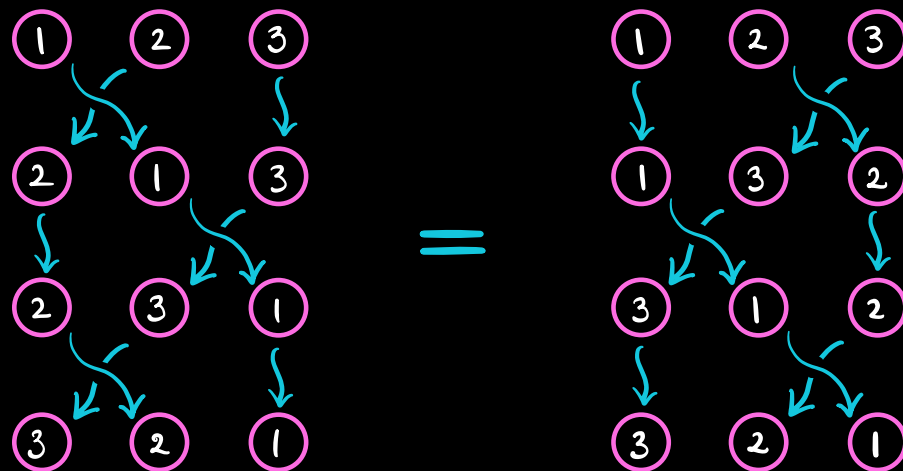
K_c = PARTICLE c AGAINST BOUNDARY

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY

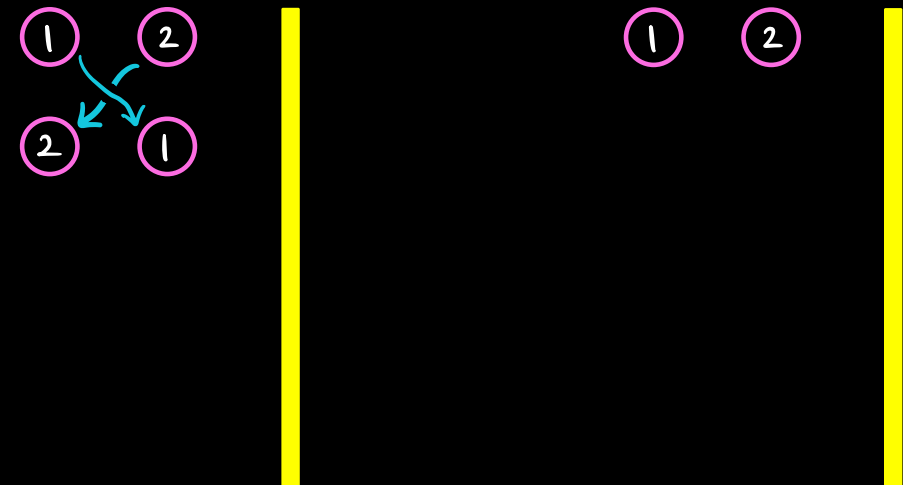


R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY



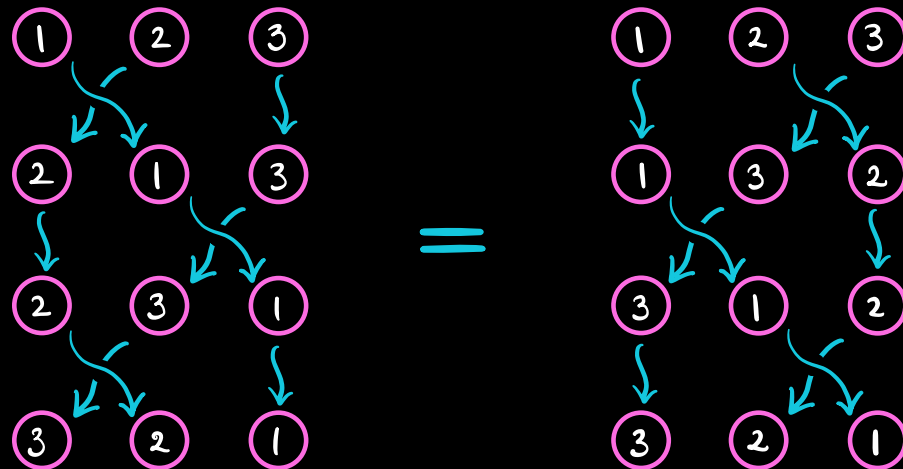
K_c = PARTICLE c AGAINST BOUNDARY

R_{12}

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↪ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



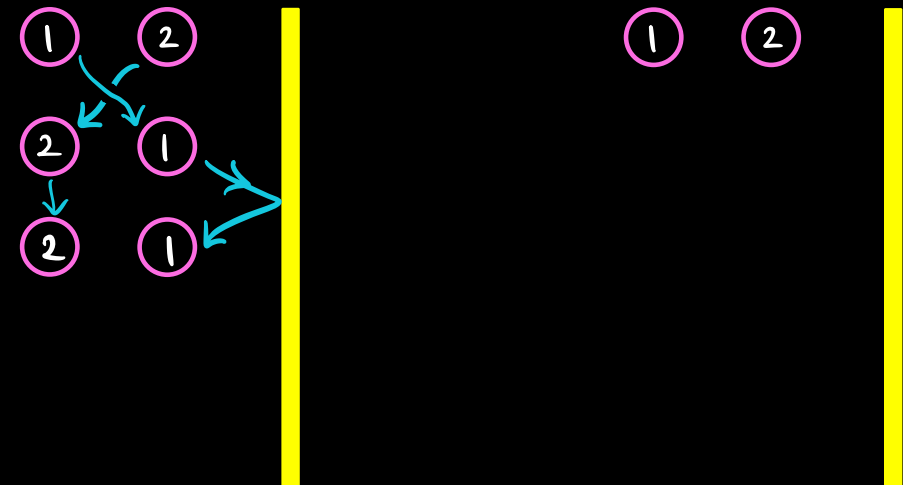
R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES ↪
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY



K_c = PARTICLE c AGAINST BOUNDARY

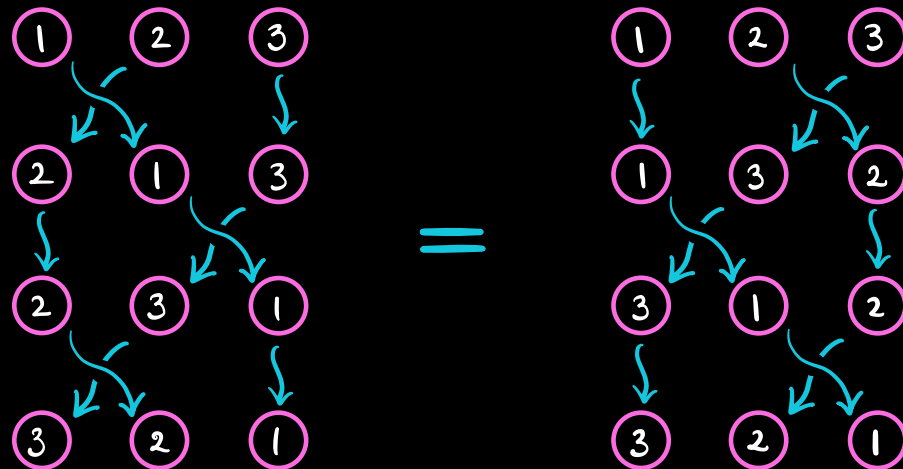
$$R_{12} K_1$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY

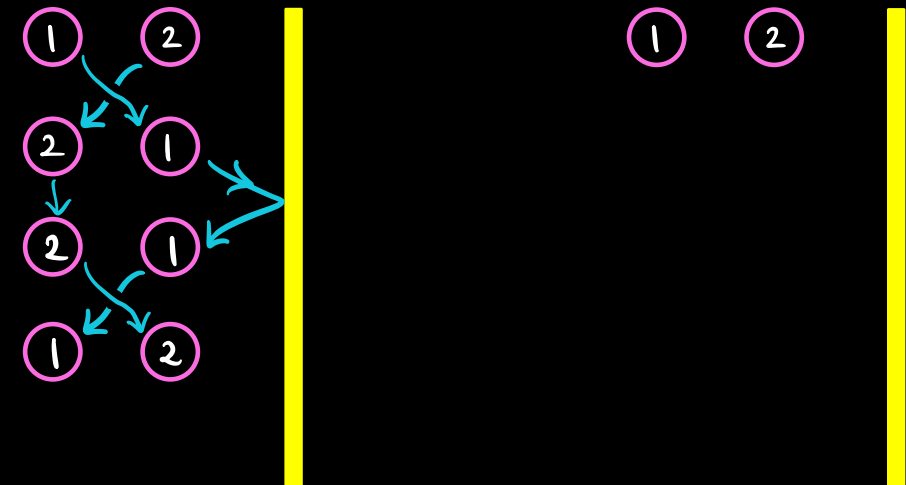


R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY



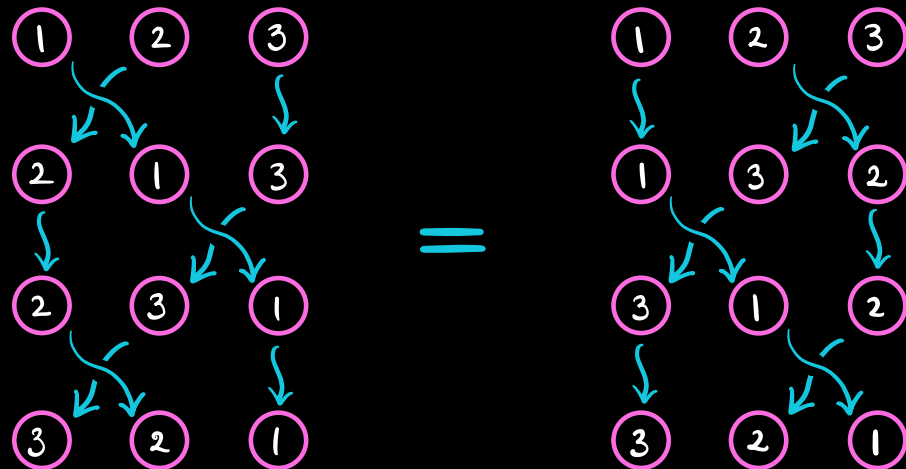
K_c = PARTICLE c AGAINST BOUNDARY

$$R_{12} K_1 R_{21}$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↪ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



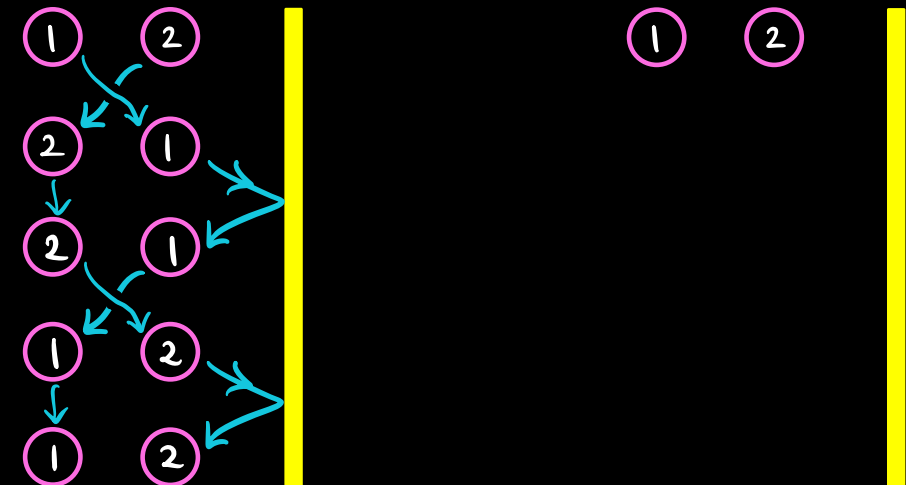
R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES ↪
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY



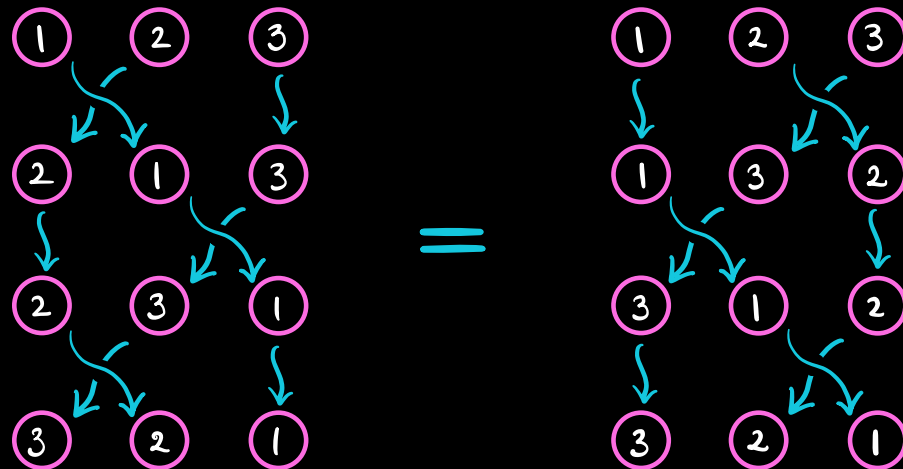
K_c = PARTICLE c AGAINST BOUNDARY

$$R_{12} K_1 R_{21} K_2$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



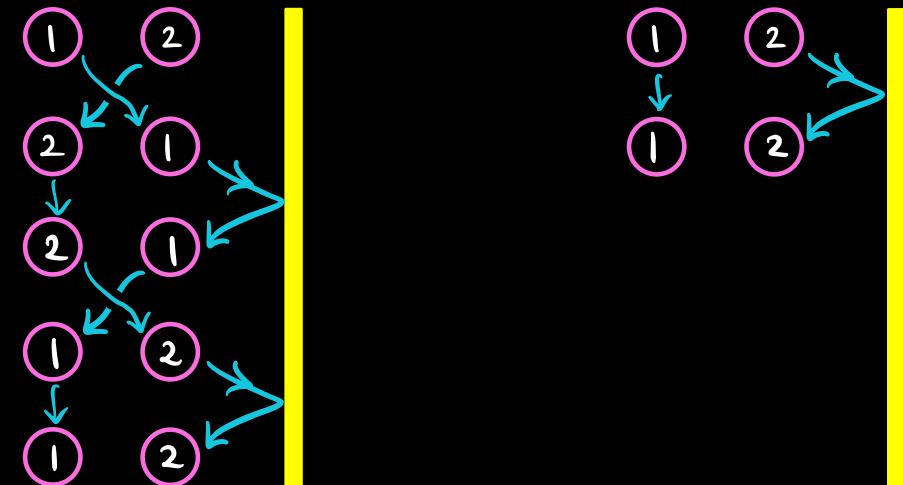
R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES
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WITH BOUNDARY



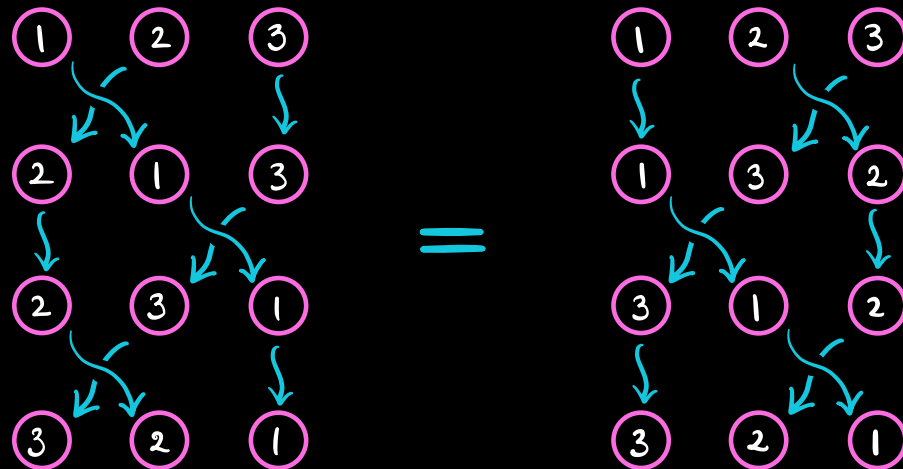
K_c = PARTICLE c AGAINST BOUNDARY

$$R_{12} K_1 R_{21} K_2 \quad K_2$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↪ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



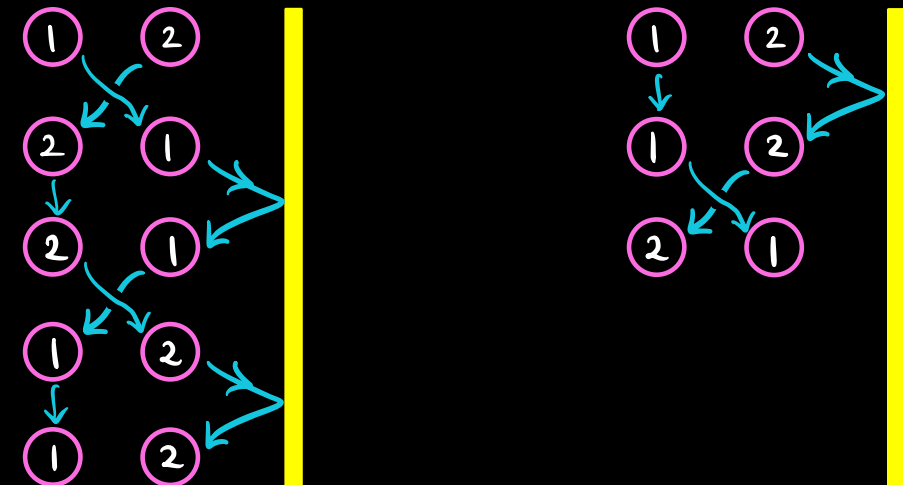
R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES ↪
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY



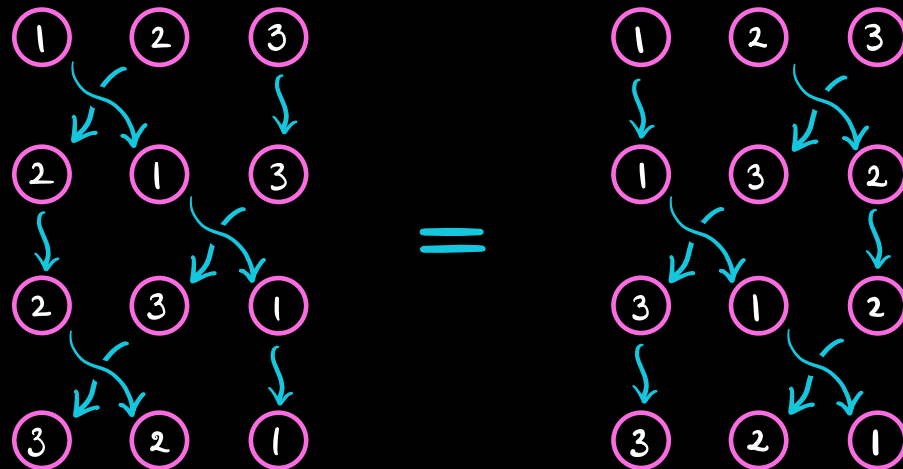
K_c = PARTICLE c AGAINST BOUNDARY

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12}$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



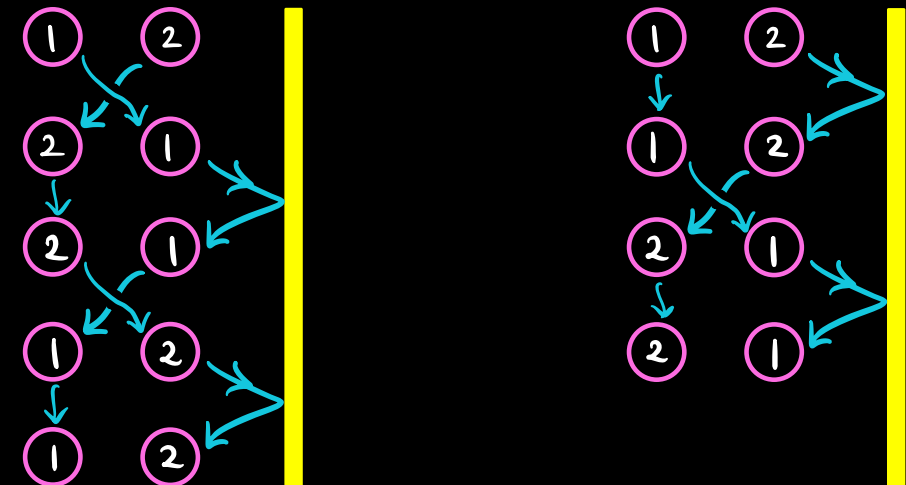
R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY



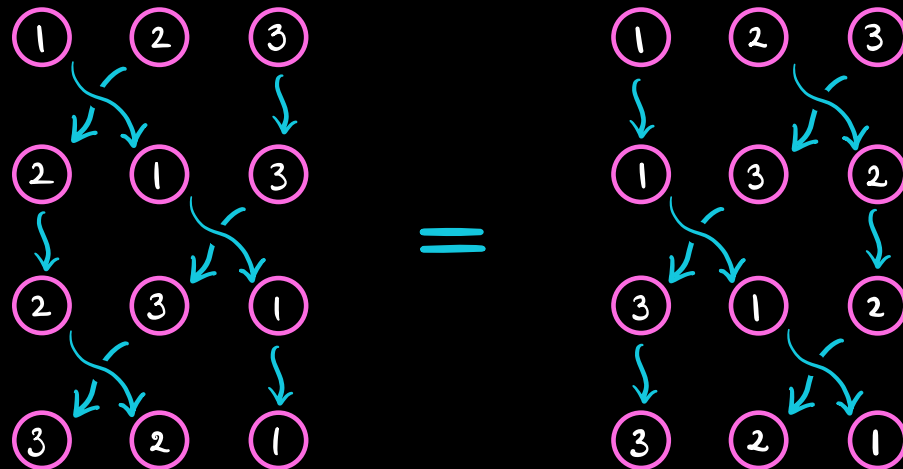
K_c = PARTICLE c AGAINST BOUNDARY

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

↪ STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



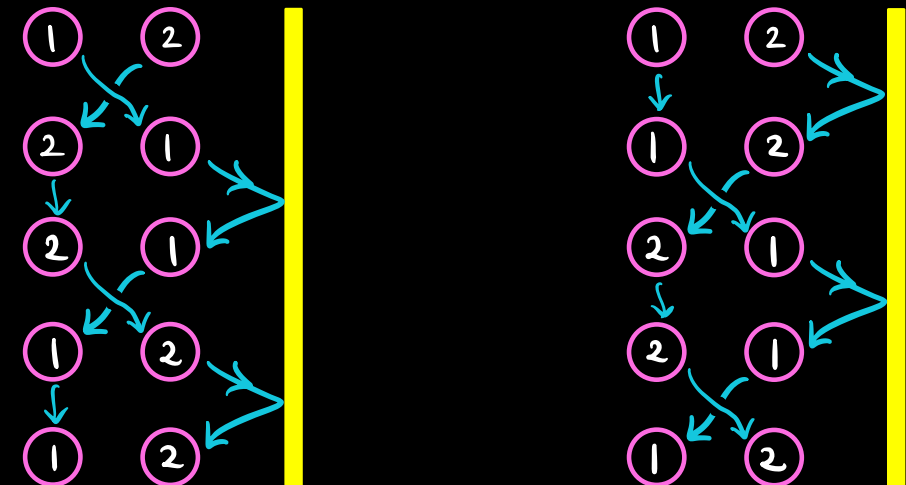
R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES ↪
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY



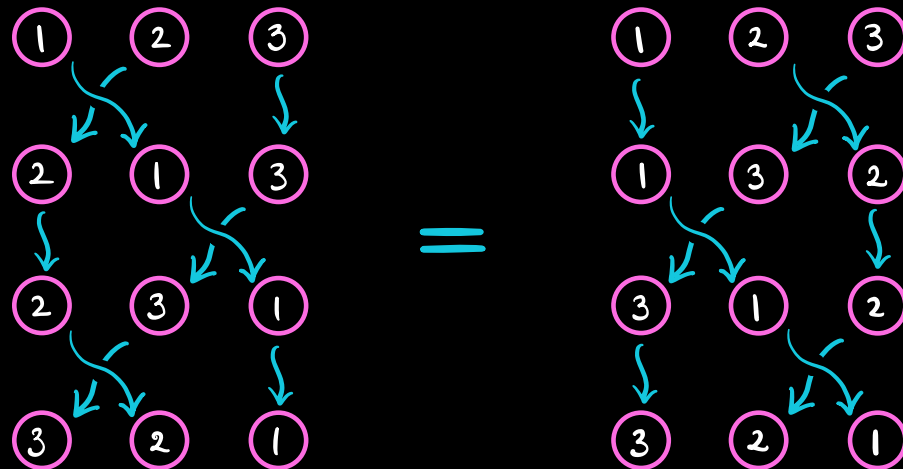
K_c = PARTICLE c AGAINST BOUNDARY

$$R_{12} K_1 R_{21} K_2 \quad K_2 R_{12} K_1 R_{21}$$

YANG (1967)
BAXTER (1972)

A TALE OF TWO EQUATIONS

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY



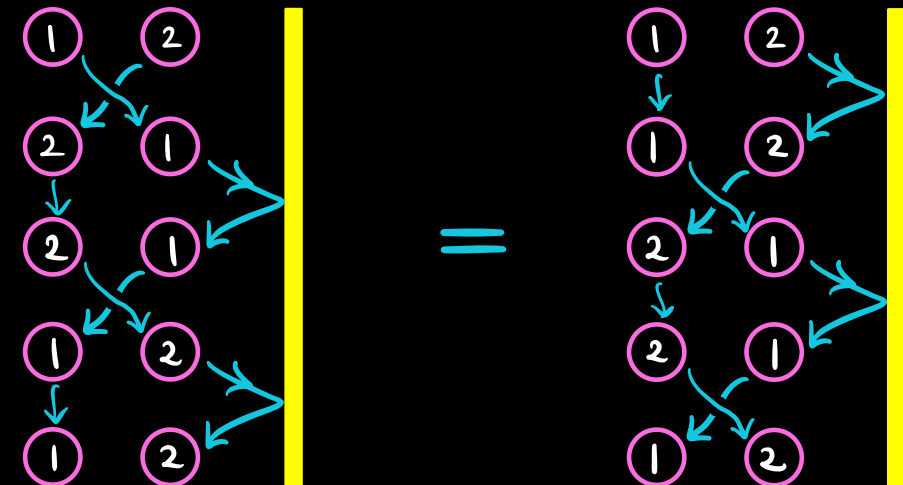
R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (QYBE)

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY



K_c = PARTICLE c AGAINST BOUNDARY

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (QRE)

YANG (1967)
BAXTER (1972)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY

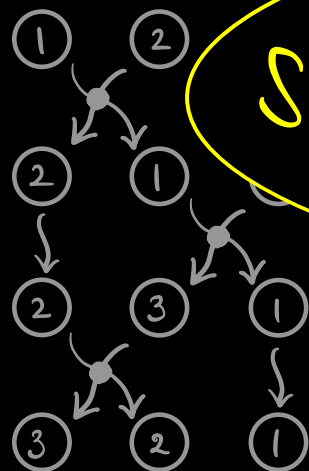
A TALE OF TWO EQUATIONS

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY

GOAL

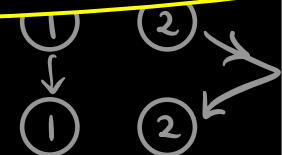
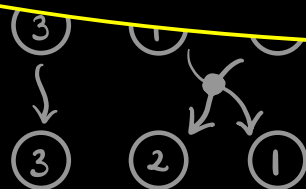
STUDY SOLUTIONS TO THESE EQUATIONS



R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

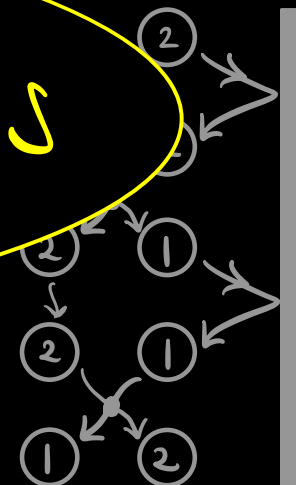
QUANTUM YANG-BAXTER EQ. (qYBE)



K_c = PARTICLE c AGAINST BOUNDARY

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)



YANG (1967)
BAXTER (1972)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITHOUT BOUNDARY

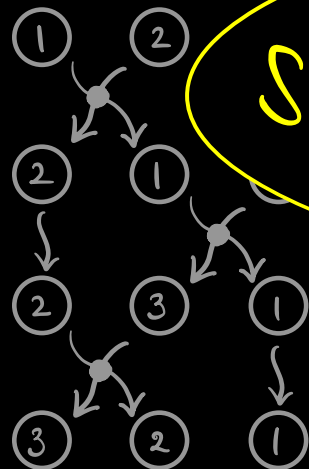
A TALE OF TWO EQUATIONS

CHEREDNIK (1984)
SKLYANIN (1988)

STUDIED HOW PARTICLES
SCATTER IN A PHYSICAL SYSTEM
WITH BOUNDARY

GOAL

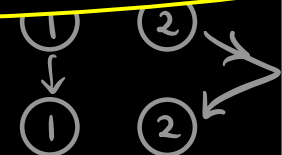
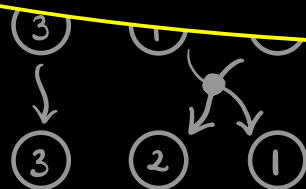
STUDY SOLUTIONS TO THESE EQUATIONS
ESPECIALLY THIS ONE



R_{ab} = PARTICLE a PASS PARTICLE b

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

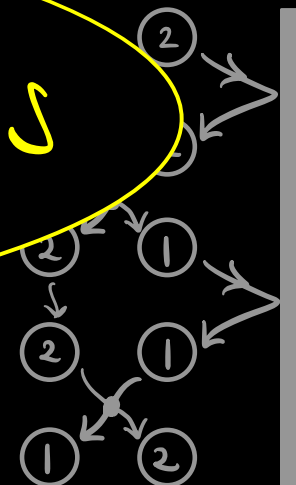
QUANTUM YANG-BAXTER EQ. (qYBE)



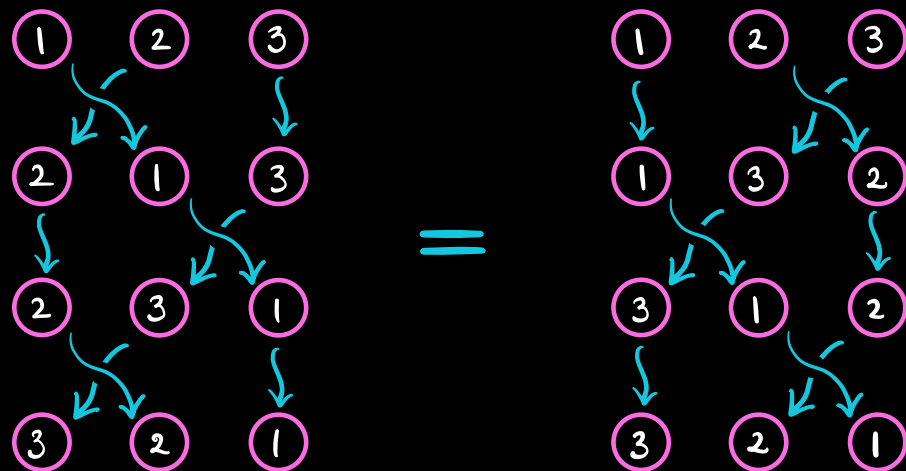
K_c = PARTICLE c AGAINST BOUNDARY

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

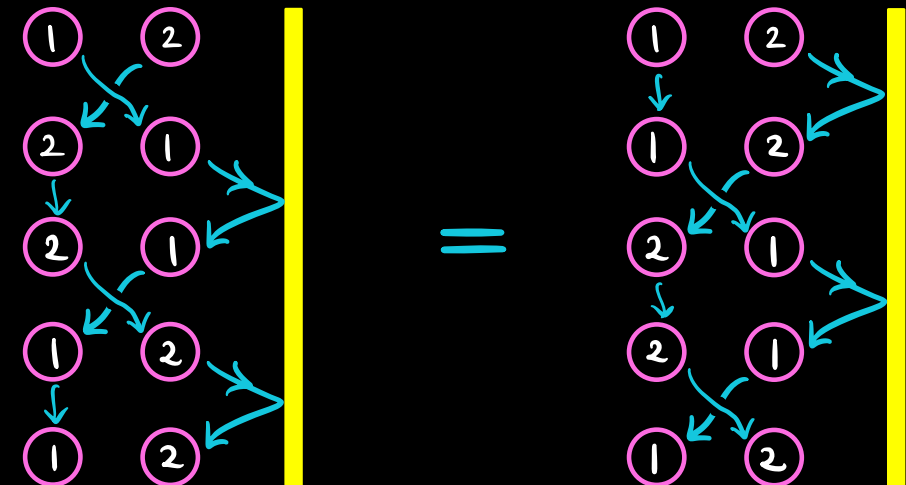


A TALE OF TWO EQUATIONS



$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

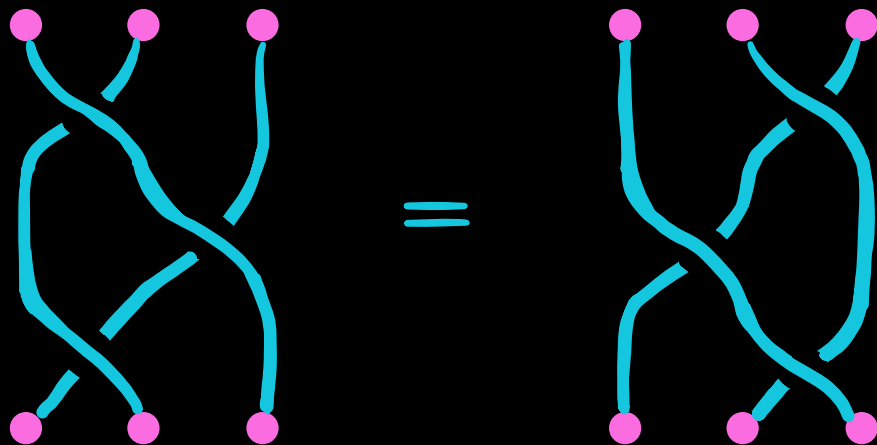
QUANTUM YANG-BAXTER EQ. (qYBE)



$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

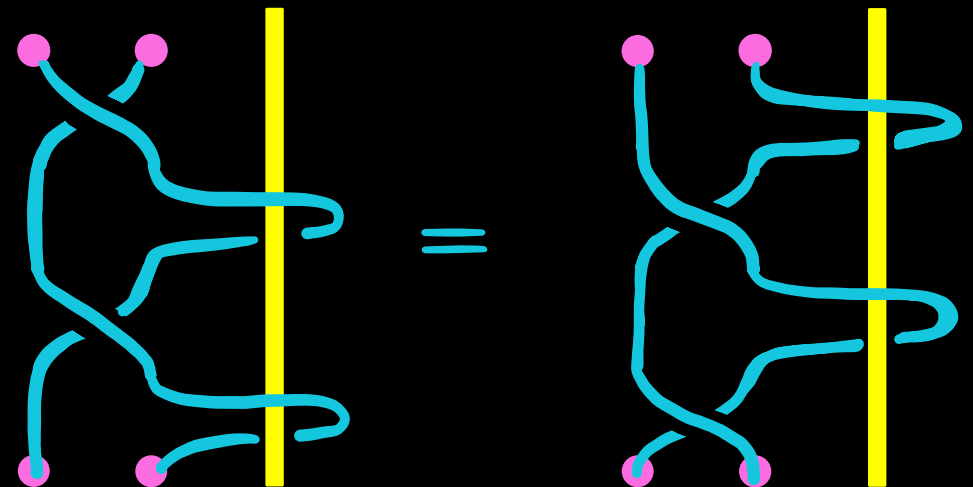
QUANTUM REFLECTION EQ (qRE)

STRING MODELS



$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

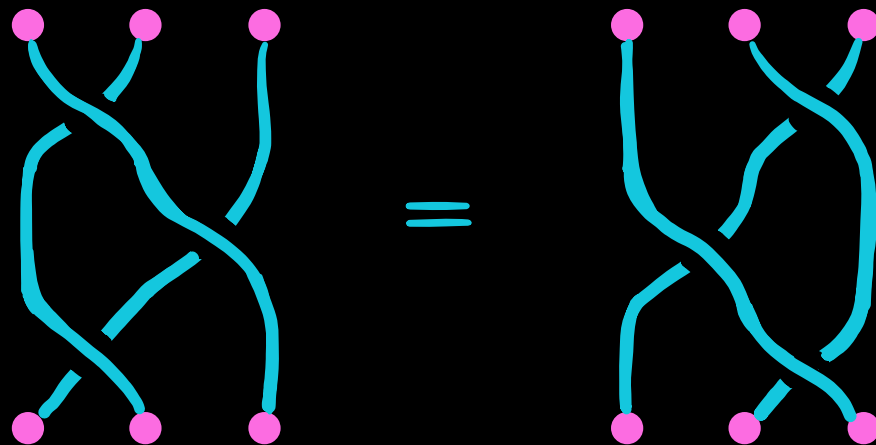
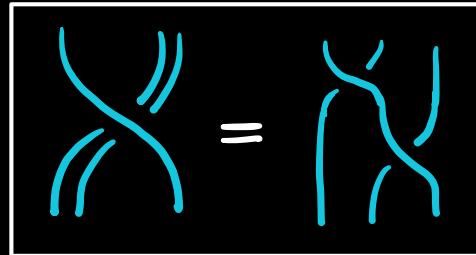
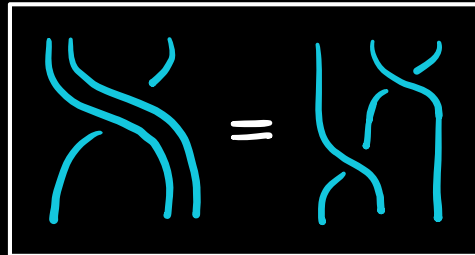


$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

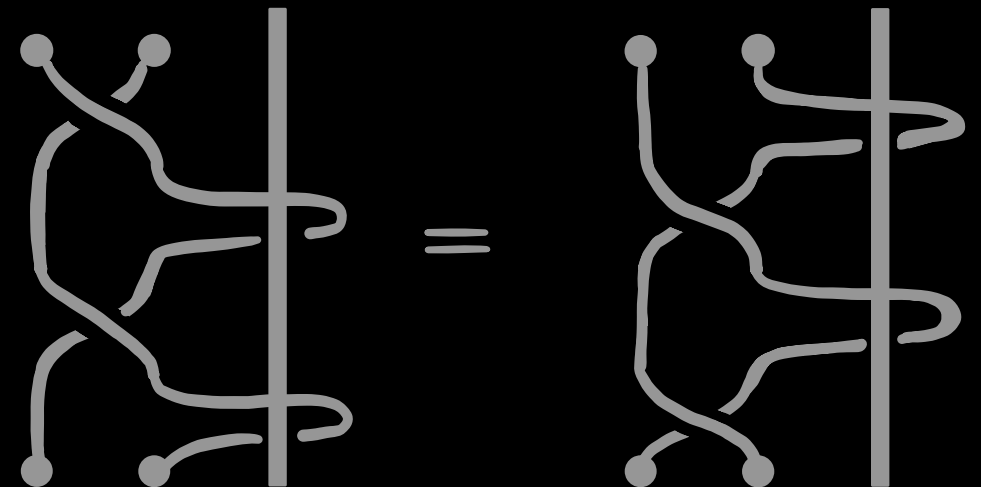
"BRAID AXIOMS"

STRING MODELS



$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

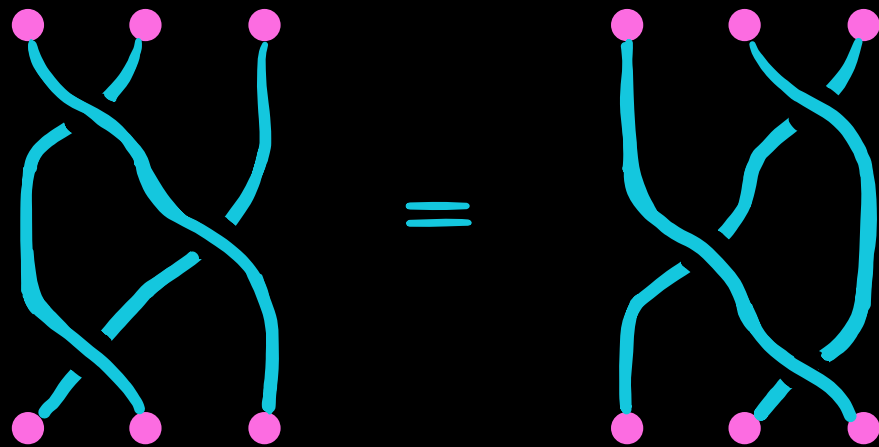
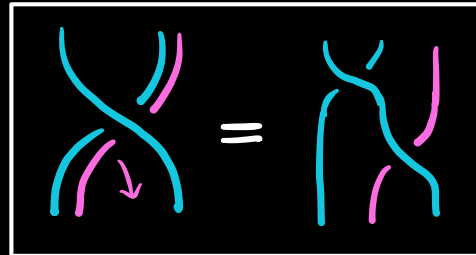
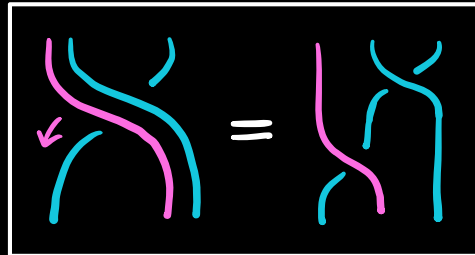


$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

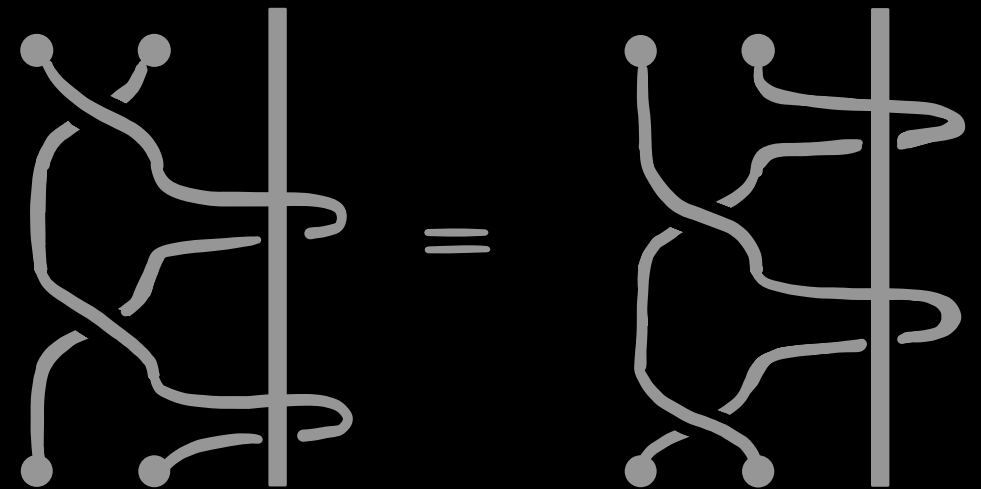
"BRAID AXIOMS"

STRING MODELS



$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

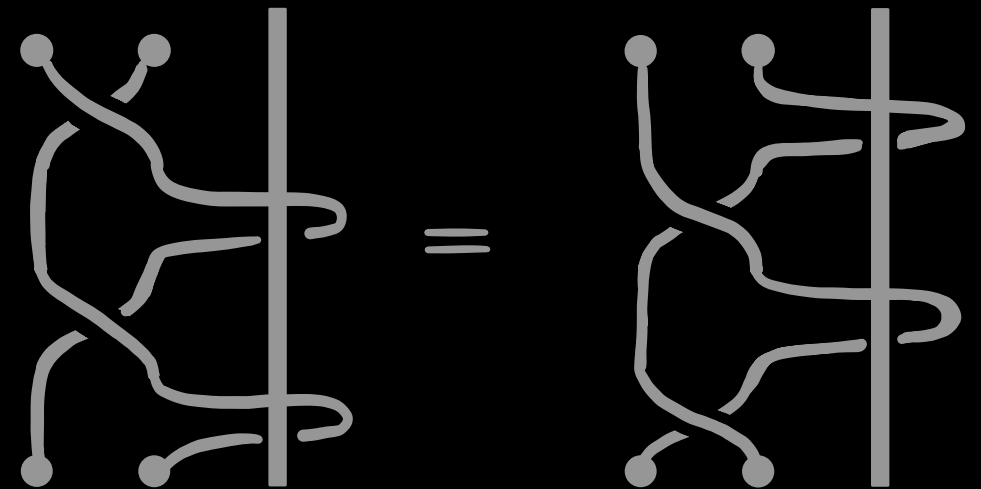
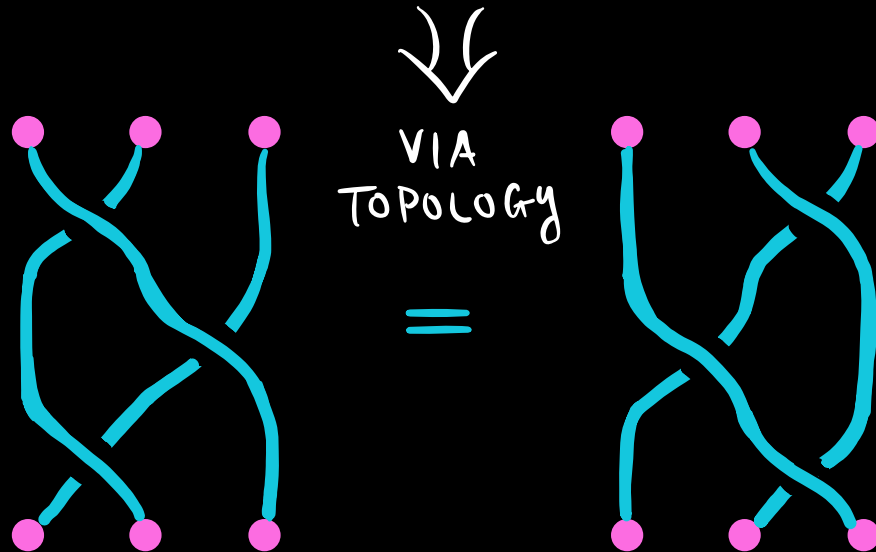
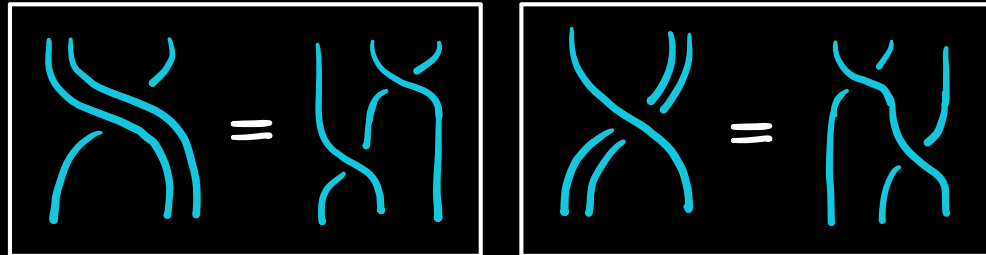


$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

"BRAID AXIOMS"

STRING MODELS

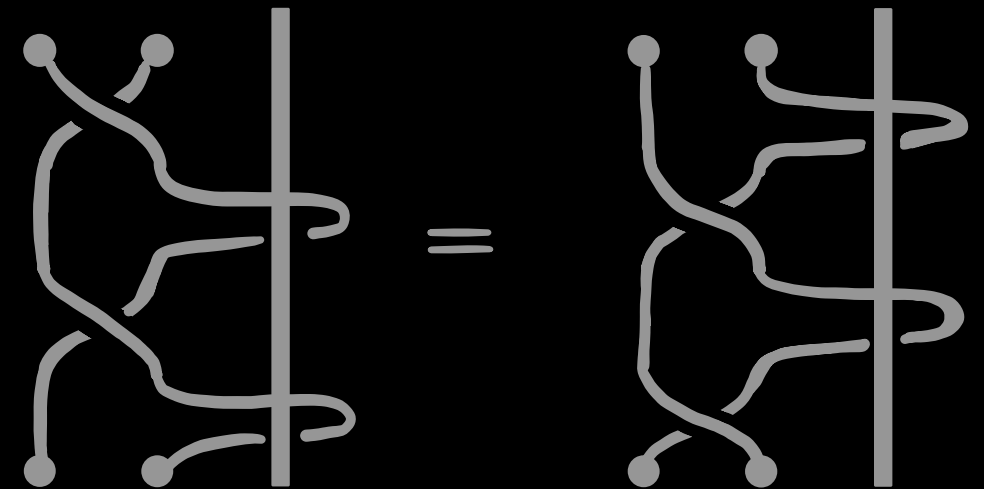
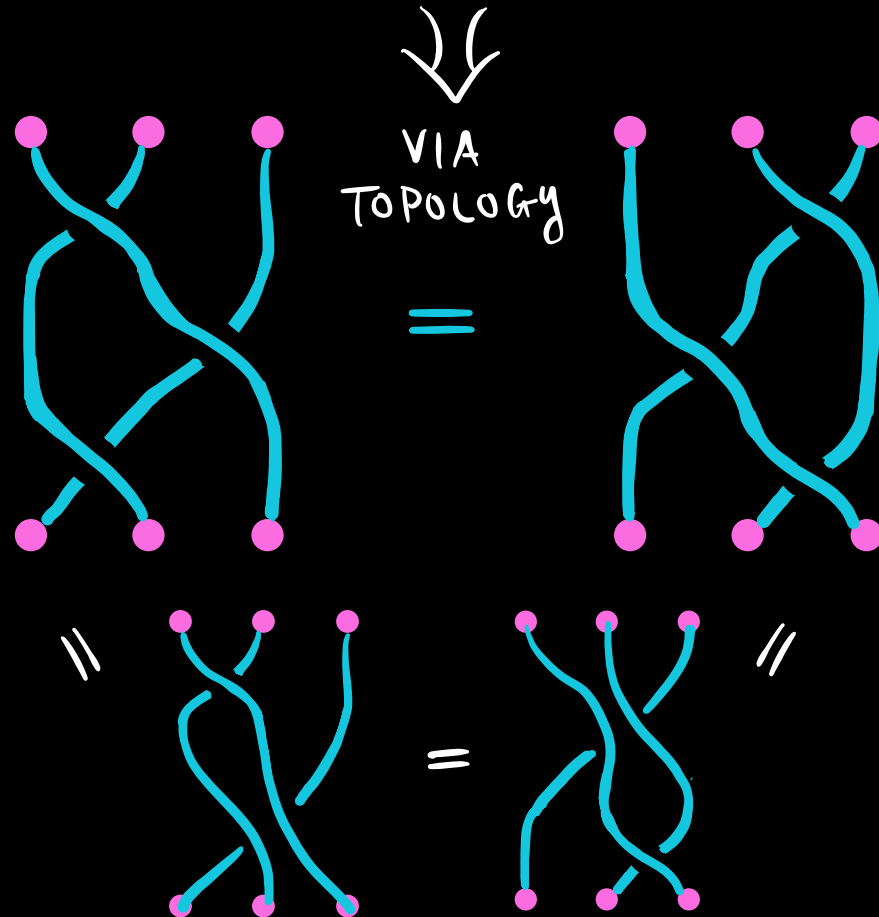
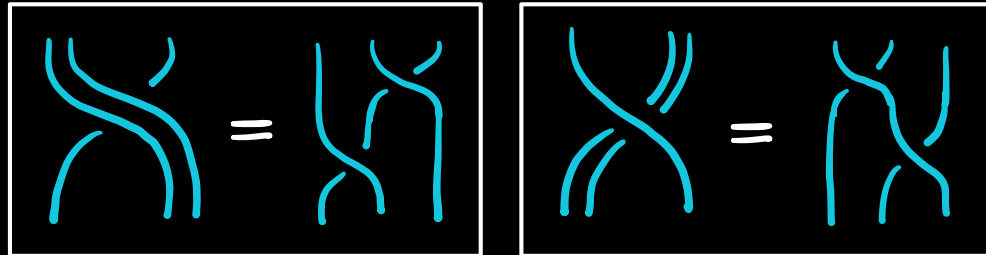


$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

"BRAID AXIOMS"

STRING MODELS



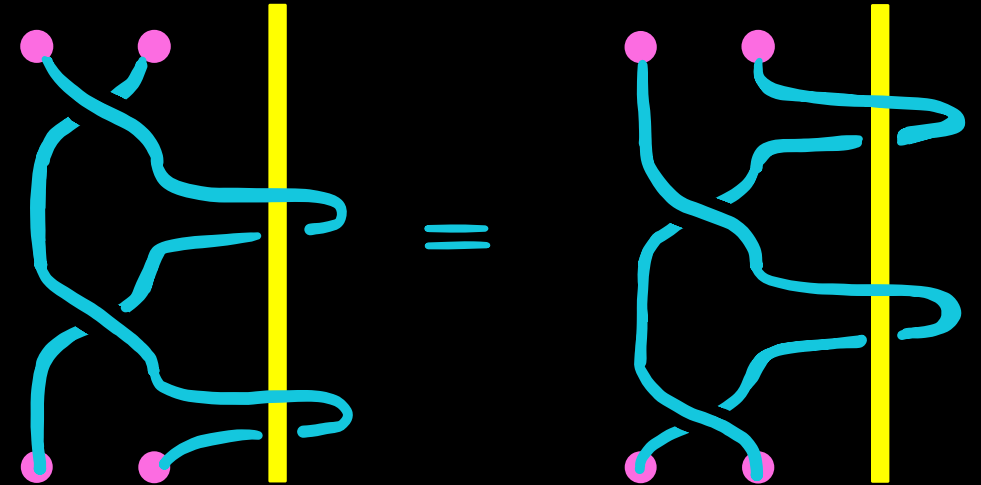
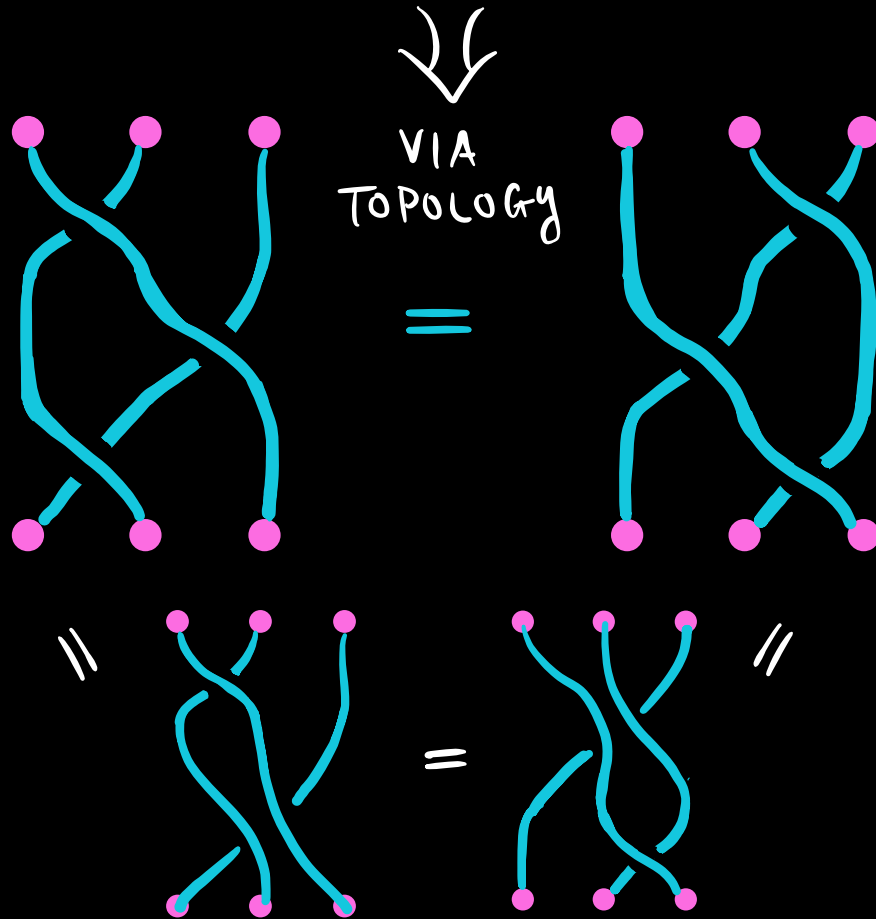
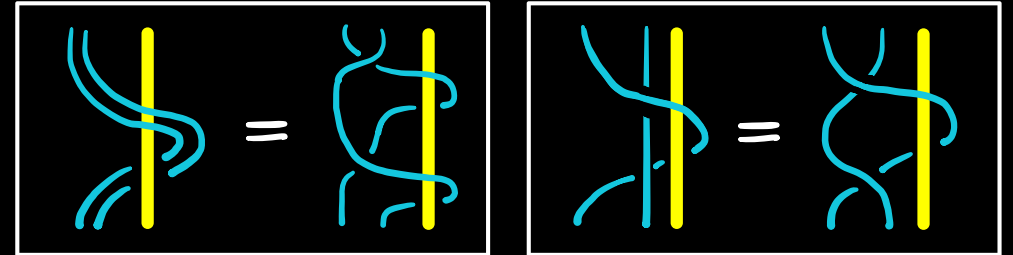
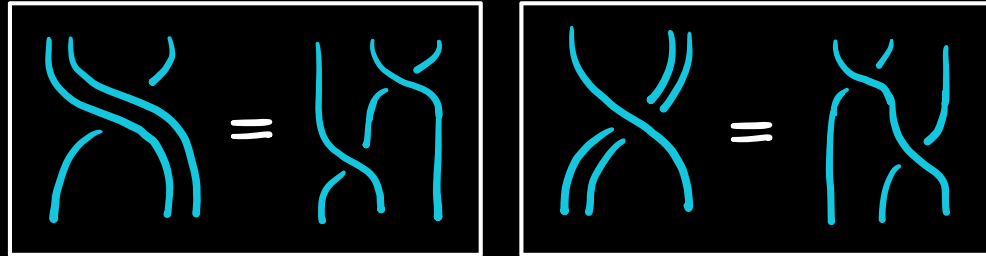
$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

"BRAID AXIOMS"

STRING MODELS

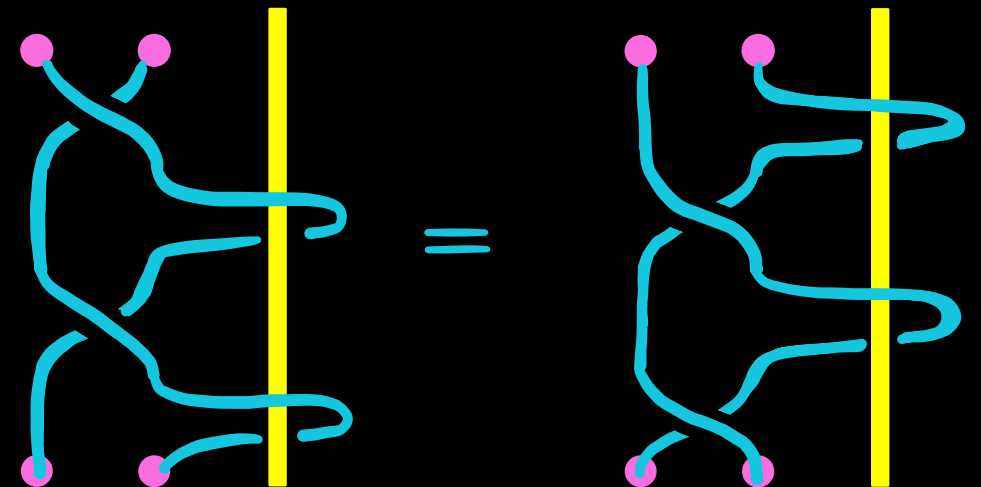
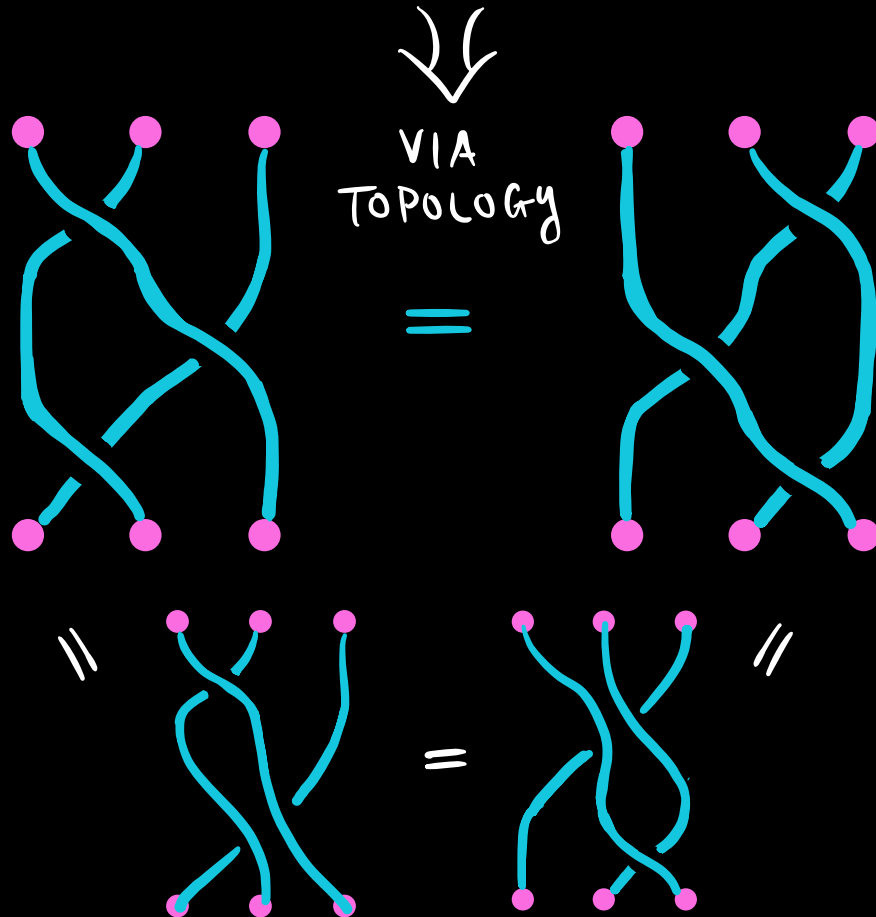
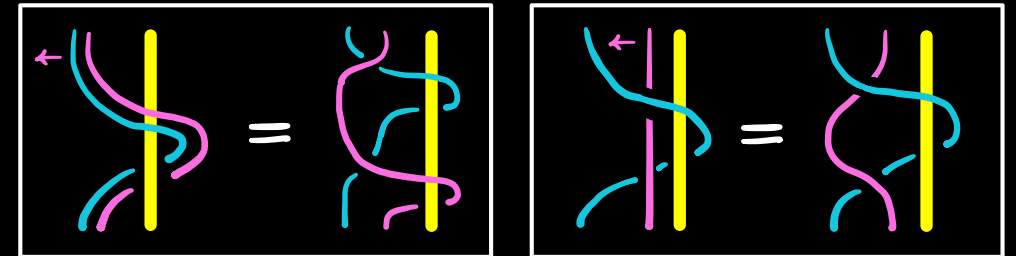
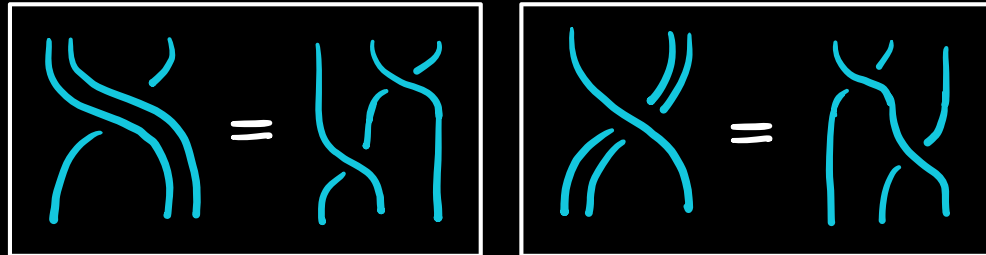
"BRAID AXIOMS"



"BRAID AXIOMS"

STRING MODELS

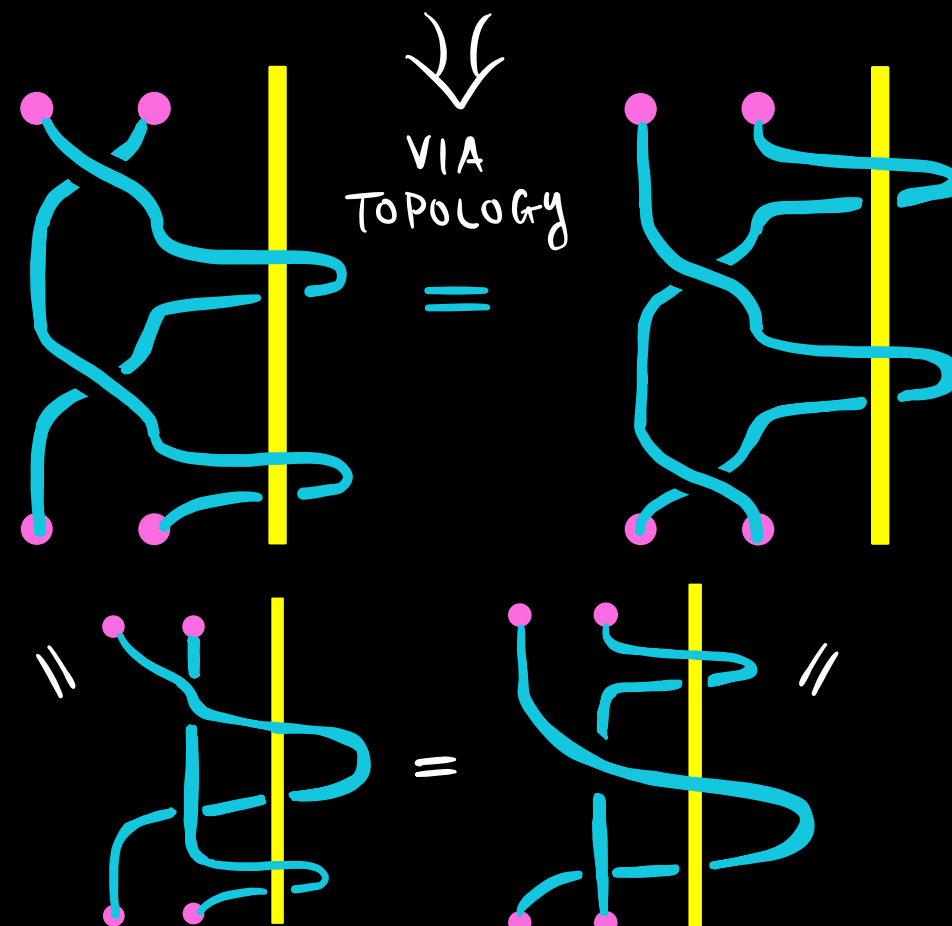
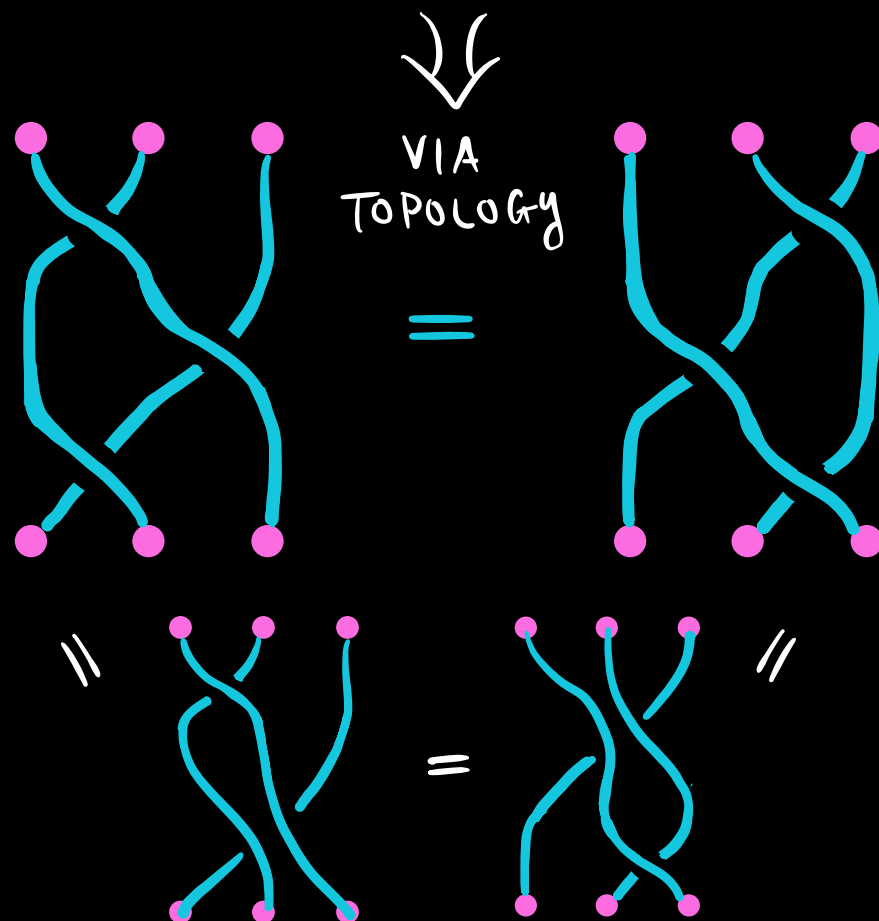
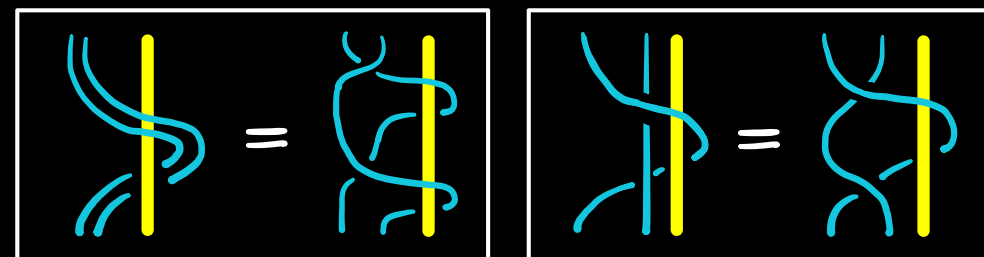
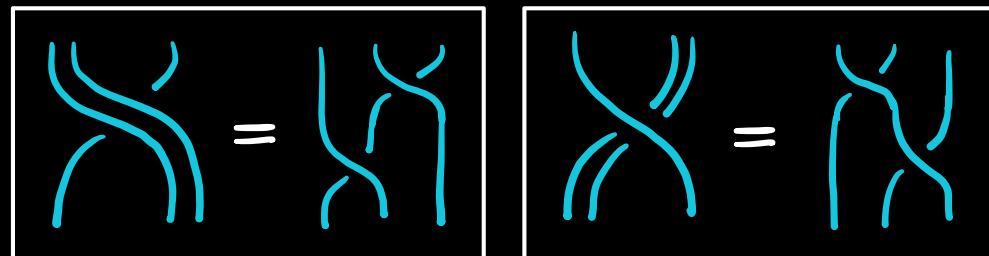
"BRAID AXIOMS"



"BRAID AXIOMS"

STRING MODELS

"BRAID AXIOMS"



"BRAID AXIOMS"

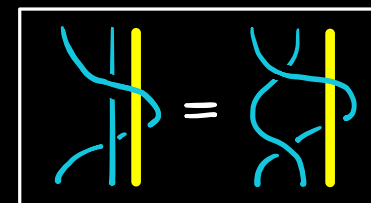
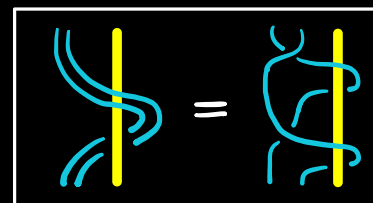
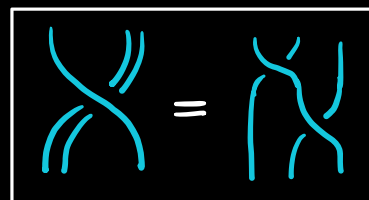
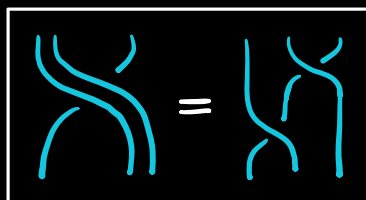
STRING MODELS

"BRAID AXIOMS"

LET'S CAPTURE THESE EQUATIONS
≡ CATEGORICALLY ≡

BRAIDED-TYPE CATEGORIES

"BRAID
AXIOMS"



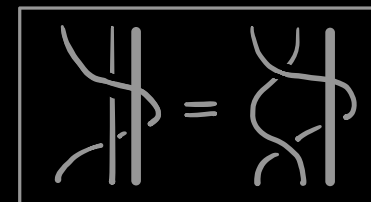
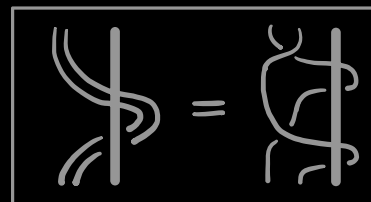
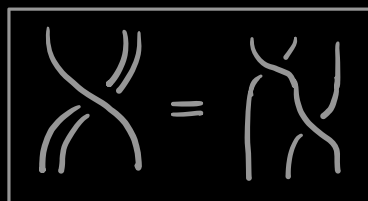
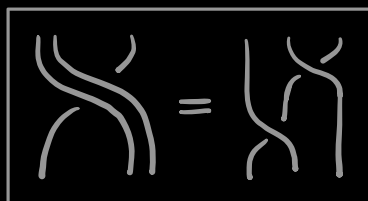
"BRAID
AXIOMS"

Joyal-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BRAIDED MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$

"BRAID
AXIOMS"



"BRAID
AXIOMS"

Joyal-Street
(1986, 1993)

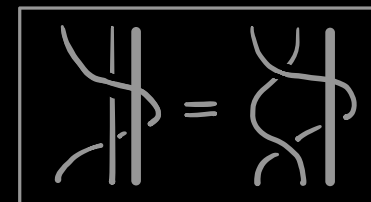
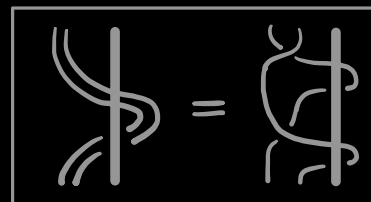
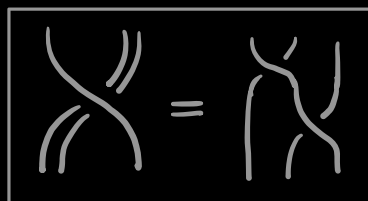
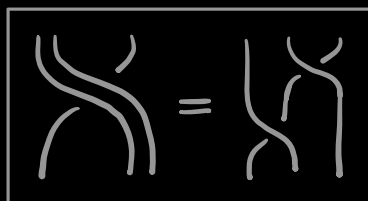
BRAIDED-TYPE CATEGORIES

BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY

"BRAID
AXIOMS"



"BRAID
AXIOMS"

Joyal-Street
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BRAIDED MONOIDAL CATEGORY

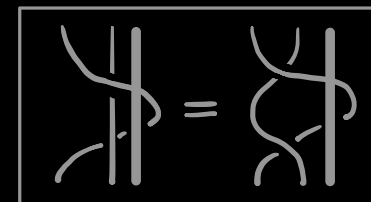
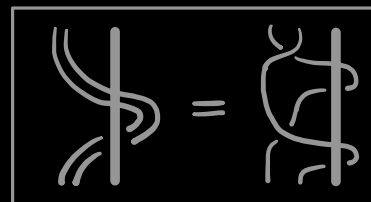
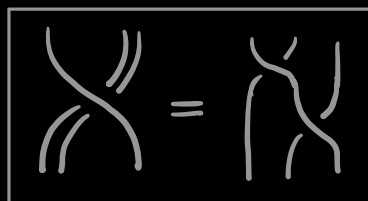
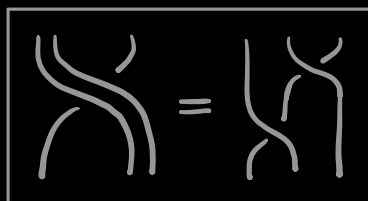
$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY

BIFUNCTOR

$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

"BRAID
AXIOMS"



"BRAID
AXIOMS"

Joyal-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY

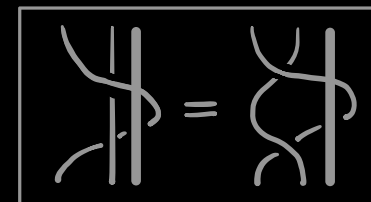
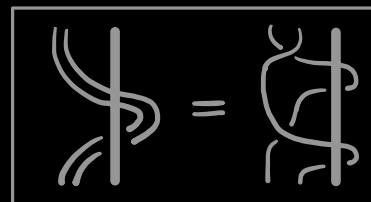
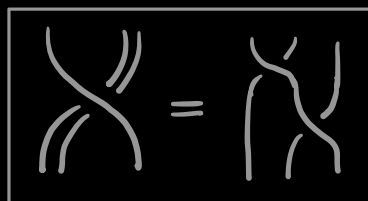
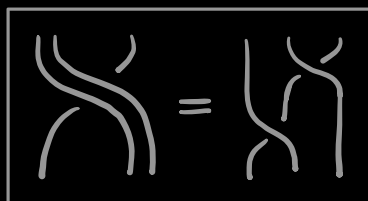
BIFUNCTOR

$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT

IN $\mathcal{C}$

"BRAID
AXIOMS"



"BRAID
AXIOMS"

Joyal-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

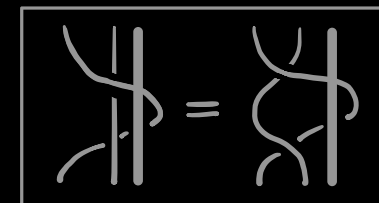
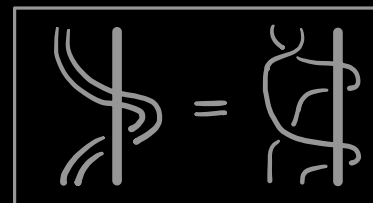
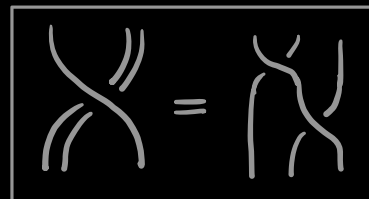
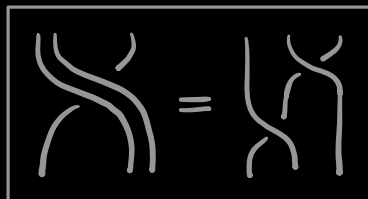
BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY BIFUNCTOR DIST. OBJECT
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ IN $\mathcal{C}$

BEHAVES LIKE A MONOID

"BRAID
AXIOMS"



"BRAID
AXIOMS"

JOYAL-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BRAIDED MONOIDAL CATEGORY

$$(C, \otimes, 1, c)$$

CATEGORY

$$f: X \rightarrow Y$$

BIFUNCTOR

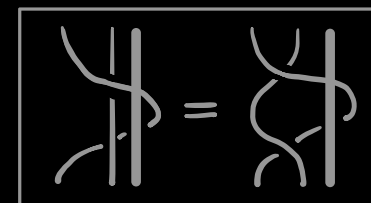
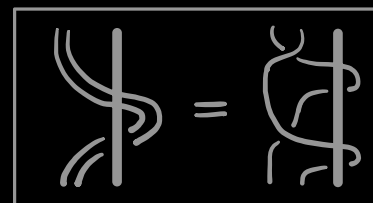
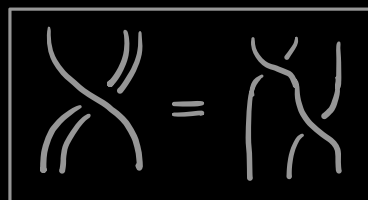
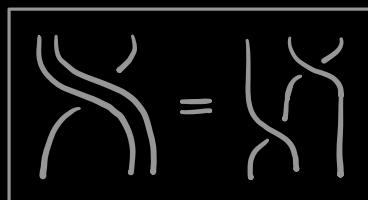
$$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

DIST. OBJECT

IN E



"BRAID AXIOMS"



"BRAID AXIOMS"

Joyal-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY

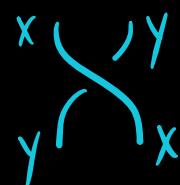
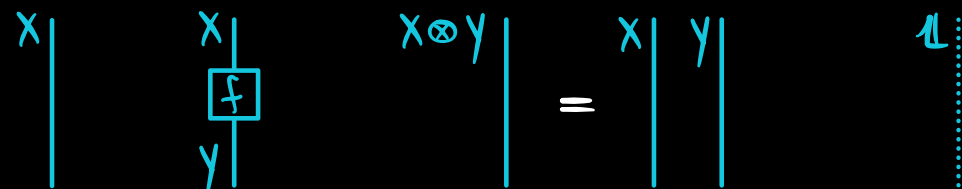
$f: x \rightarrow y$

BIFUNCTOR

$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT

IN $\mathcal{C}$

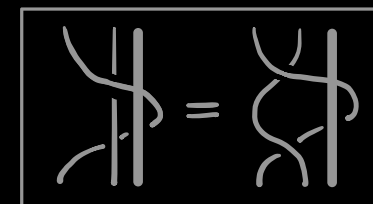
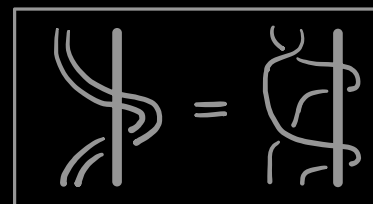
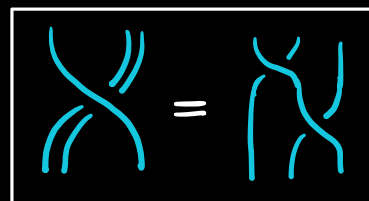
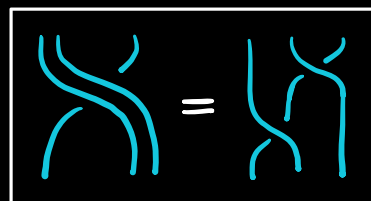


$c := \{ c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x \}_{x,y \in \mathcal{C}}$

NAT.
ISOM.

SUCH THAT

"BRAID
AXIOMS"



"BRAID
AXIOMS"

Joyal-STREET
(1986, 1993)

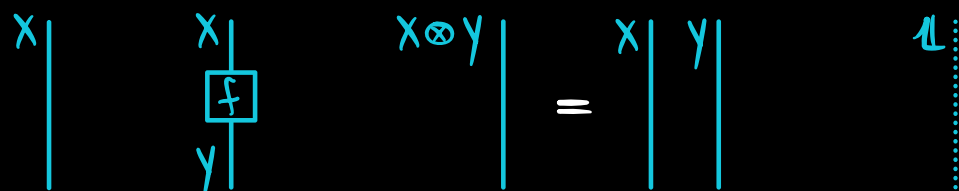
BRAIDED-TYPE CATEGORIES

BRAIDED MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY
 $f: x \rightarrow y$

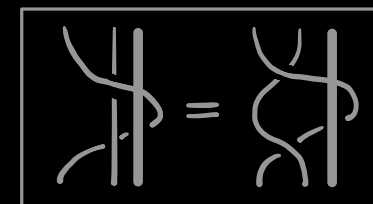
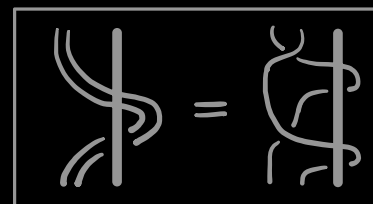
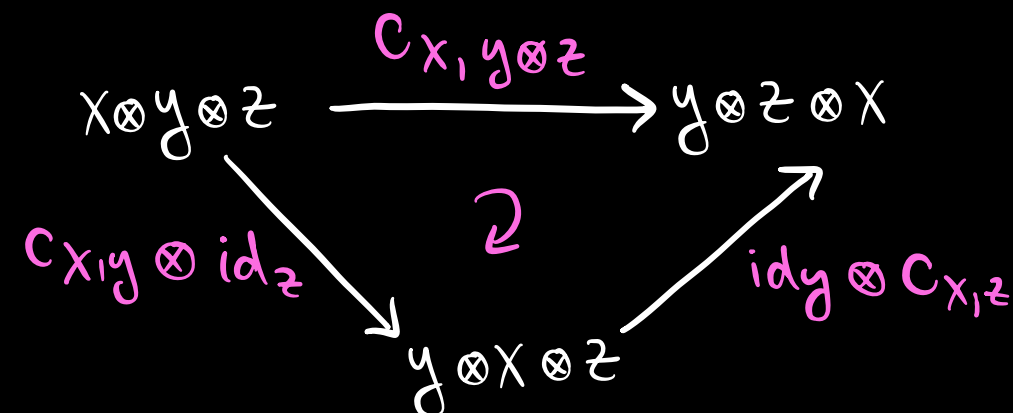
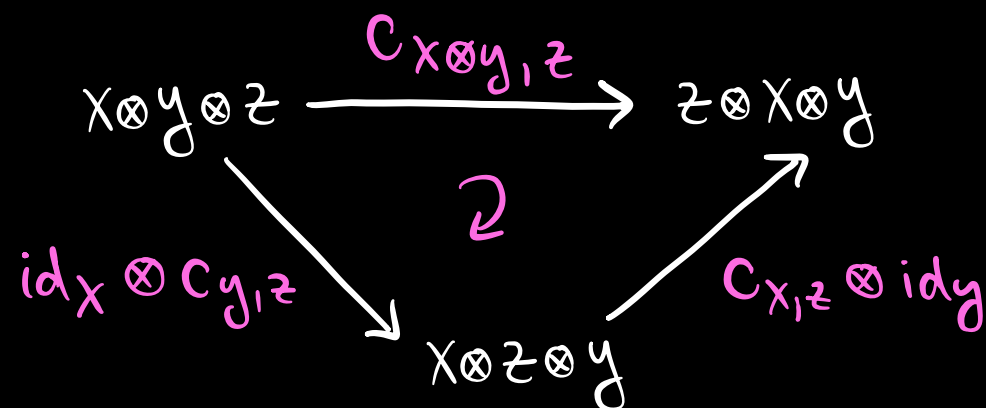
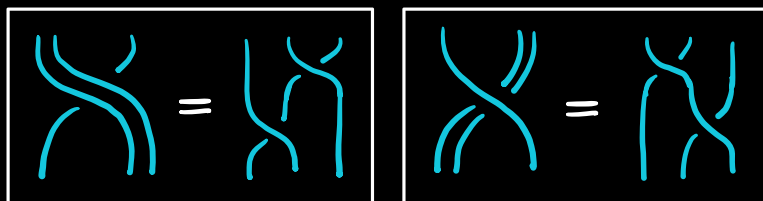
BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT
IN $\mathcal{C}$



$c := \{ c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x \}_{x,y \in \mathcal{C}}$
NAT. ISOM.
SUCH THAT

"BRAID AXIOMS"



"BRAID AXIOMS"

Joyal-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BRAIDED MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY

$$f: x \rightarrow y$$

BIFUNCTOR

$$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

DIST. OBJECT

IN $\mathcal{C}$

$$\begin{array}{c} x \\ | \\ y \end{array} = \begin{array}{c} x \\ | \\ \boxed{f} \\ | \\ y \end{array} \quad \begin{array}{c} x \otimes y \\ | \\ \mathbb{1} \end{array} = \begin{array}{c} x \\ | \\ y \end{array}$$

$$\begin{array}{c} x \\ \swarrow \searrow \\ y \end{array}$$

$$c := \{ c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x \}_{x,y \in \mathcal{C}}$$

NAT. ISOM.

SUCH THAT

"BRAID AXIOMS"

$$\begin{array}{c} \text{Braid 1} \\ \text{Diagram 1} \end{array} = \begin{array}{c} \text{Braid 2} \\ \text{Diagram 2} \end{array}$$

$$\begin{array}{c} \text{Braid 3} \\ \text{Diagram 3} \end{array} = \begin{array}{c} \text{Braid 4} \\ \text{Diagram 4} \end{array}$$

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x \otimes y, z}} & z \otimes x \otimes y \\ & \searrow \text{id}_x \otimes c_{y,z} & \nearrow c_{x,z} \otimes \text{id}_y \\ & x \otimes z \otimes y & \end{array}$$

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x,y \otimes z}} & y \otimes z \otimes x \\ & \searrow c_{x,y} \otimes \text{id}_z & \nearrow \text{id}_y \otimes c_{x,z} \\ & y \otimes x \otimes z & \end{array}$$

$$\begin{array}{c} \text{Braid 5} \\ \text{Diagram 5} \end{array} = \begin{array}{c} \text{Braid 6} \\ \text{Diagram 6} \end{array}$$

$$\begin{array}{c} \text{Braid 7} \\ \text{Diagram 7} \end{array} = \begin{array}{c} \text{Braid 8} \\ \text{Diagram 8} \end{array}$$

"BRAID AXIOMS"

JOYAL-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BROCHIER (2013)
FIX $(\mathcal{C}, \otimes, \mathbb{1}, c)$

BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY

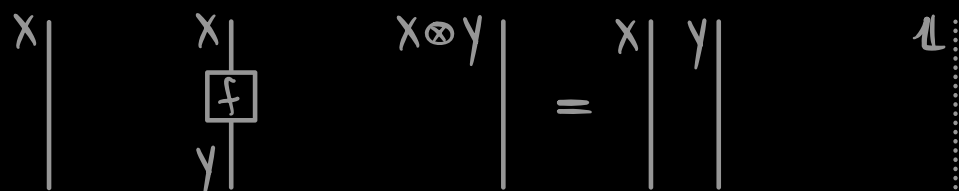
$f: x \rightarrow y$

BIFUNCTOR

$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

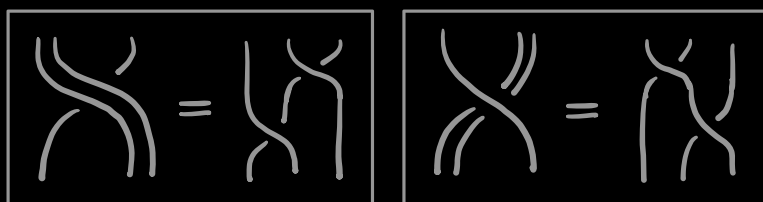
DIST. OBJECT

IN $\mathcal{C}$



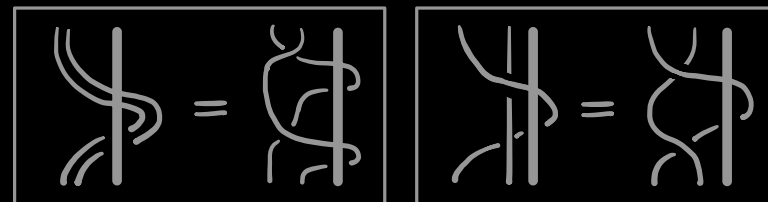
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NAT. ISOM. SUCH THAT

"BRAID AXIOMS"



BRAIDED $\mathcal{C}$ -MODULE CATEGORY

$(\mathcal{M}, \triangleright, e)$



"BRAID AXIOMS"

JOYAL-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BROCHIER (2013)
FIX $(\mathcal{C}, \otimes, \mathbb{1}, c)$

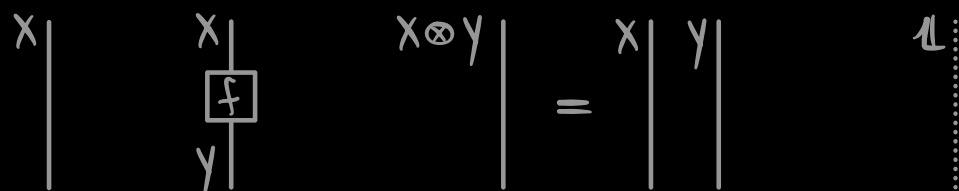
BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY
 $f: x \rightarrow y$

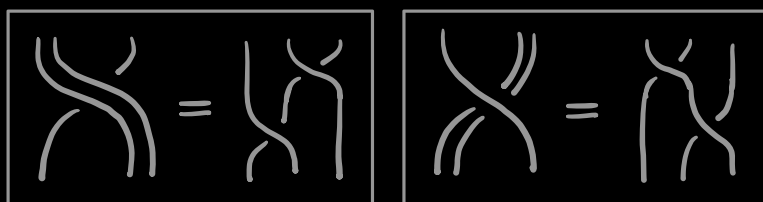
BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT
IN $\mathcal{C}$



$c := \{ c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x \}_{x,y \in \mathcal{C}}$
NAT. ISOM. SUCH THAT

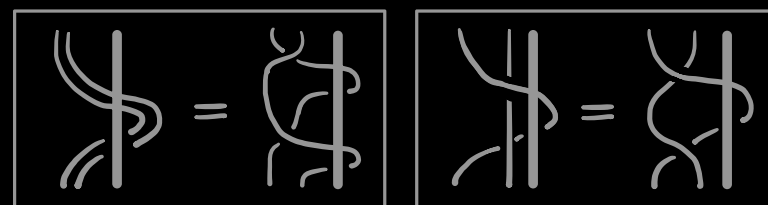
"BRAID AXIOMS"



BRAIDED $\mathcal{C}$ -MODULE CATEGORY

$(\mathcal{M}, \triangleright, e)$

CATEGORY



"BRAID AXIOMS"

JOYAL-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BROCHIER (2013)
FIX $(\mathcal{C}, \otimes, \mathbb{1}, c)$

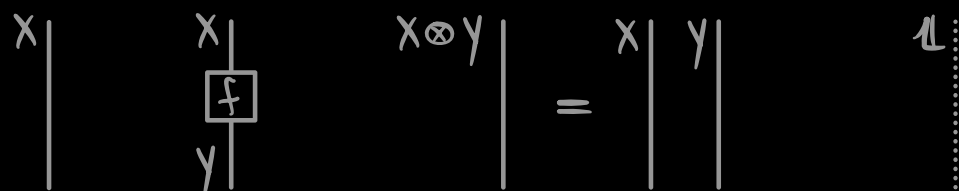
BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY
 $f: x \rightarrow y$

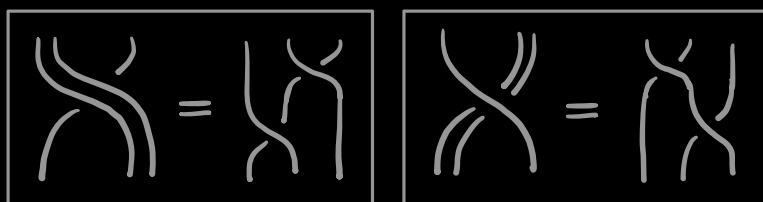
BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT
IN $\mathcal{C}$



$\mathcal{C} := \{ c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x \}_{x,y \in \mathcal{C}}$
NAT. ISOM. SUCH THAT

"BRAID AXIOMS"

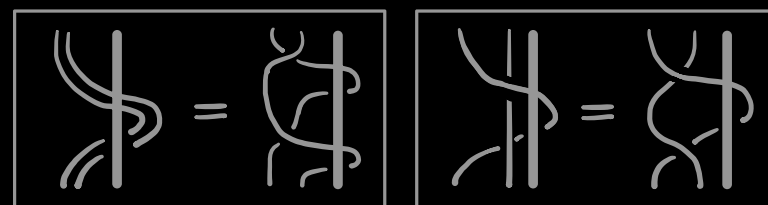


BRAIDED $\mathcal{C}$ -MODULE CATEGORY

$(\mathcal{M}, \triangleright, e)$

CATEGORY

BIFUNCTOR $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$



"BRAID AXIOMS"

JOYAL-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BROCHIER (2013)
FIX $(\mathcal{C}, \otimes, \mathbb{1}, c)$

BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY
 $f: x \rightarrow y$

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT
IN $\mathcal{C}$

$$\begin{array}{c} x \\ | \\ \hline \\ y \end{array} \quad \begin{array}{c} x \\ | \\ \boxed{f} \\ | \\ y \end{array} \quad \begin{array}{c} x \otimes y \\ | \\ \hline \\ \end{array} = \begin{array}{c} x \\ | \\ \hline \\ y \end{array} \quad \begin{array}{c} \mathbb{1} \\ \vdots \\ \hline \\ \vdots \end{array}$$

$c := \{ c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x \}_{x,y \in \mathcal{C}}$
NAT. ISOM.
SUCH THAT

"BRAID
AXIOMS"

$$\begin{array}{c} \text{Braid 1} \\ \text{Braid 2} \end{array} = \begin{array}{c} \text{Braid 1} \\ \text{Braid 2} \end{array}$$

BRAIDED $\mathcal{C}$ -MODULE CATEGORY

$(\mathcal{M}, \triangleright, e)$

CATEGORY

BIFUNCTOR $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$

BEHAVES LIKE A MODULE
OVER A MONOID

$$\begin{array}{c} \text{Braid 1} \\ \text{Braid 2} \end{array} = \begin{array}{c} \text{Braid 1} \\ \text{Braid 2} \end{array}$$

"BRAID
AXIOMS"

JOYAL-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BROCHIER (2013)
FIX ($\mathcal{C}, \otimes, \mathbb{1}, c$)

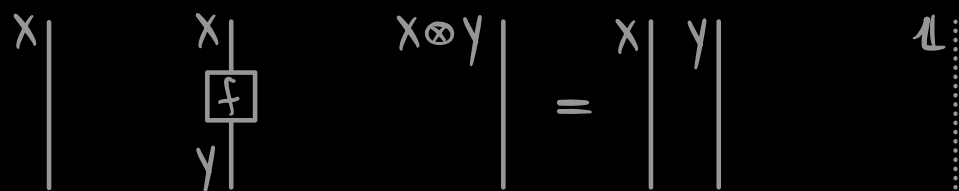
BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY
 $f: x \rightarrow y$

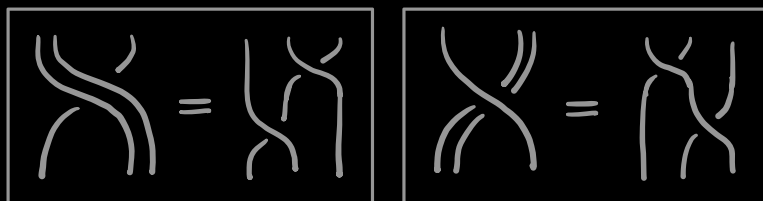
BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT
IN $\mathcal{C}$



$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
NAT. ISOM.
SUCH THAT

"BRAID AXIOMS"

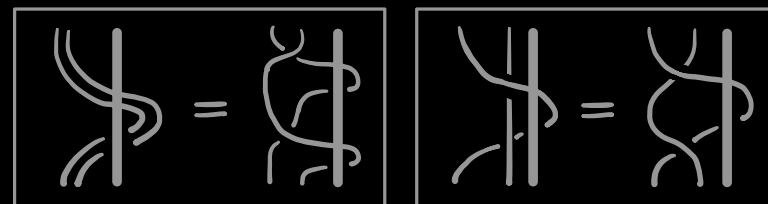


BRAIDED $\mathcal{C}$ -MODULE CATEGORY

$(\mathcal{M}, \triangleright, e)$

CATEGORY

BIFUNCTOR $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$



"BRAID AXIOMS"

JOYAL-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BROCHIER (2013)
FIX $(\mathcal{C}, \otimes, \mathbb{1}, c)$

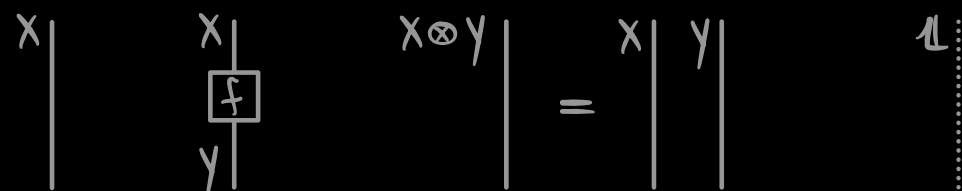
BRAIDED MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

CATEGORY
 $f: x \rightarrow y$

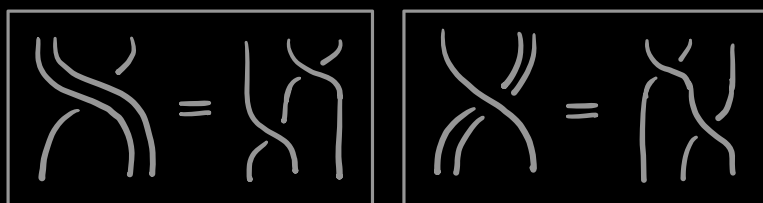
BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT
IN $\mathcal{C}$



$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
NAT. ISOM.
SUCH THAT

"BRAID AXIOMS"



BRAIDED $\mathcal{C}$ -MODULE CATEGORY

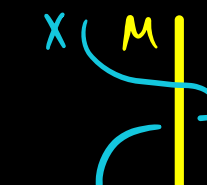
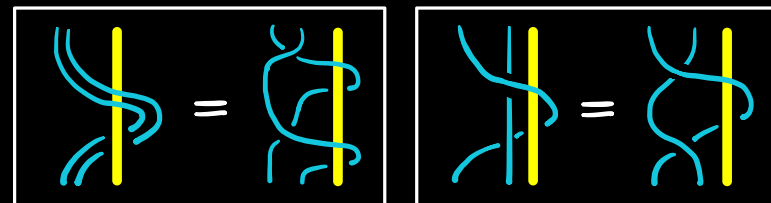
$(\mathcal{M}, \triangleright, e)$

CATEGORY

BIFUNCTOR $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$



$e := \{e_{x,M} : x \triangleright M \xrightarrow{\sim} x \triangleright M\}_{x \in \mathcal{C}, M \in \mathcal{M}}$
NAT. ISOM.
SUCH THAT



"BRAID AXIOMS"

BRAIDED-TYPE CATEGORIES

BROCHIER (2013)
 $\text{FIX}(\mathcal{C}, \otimes, \mathbb{1}, c)$

$$\begin{array}{ccc}
 X \otimes Y \triangleright M & \xrightarrow{e_{X \otimes Y, M}} & X \otimes Y \triangleright M \\
 \downarrow c_{Y, X}^{-1} \triangleright \text{id}_M & \searrow & \uparrow \text{id}_X \otimes e_{Y, M} \\
 Y \otimes X \triangleright M & & X \otimes Y \triangleright M \\
 \downarrow \text{id}_Y \otimes e_{X, M} & \nearrow & \uparrow c_{Y, X} \triangleright \text{id}_M \\
 Y \otimes X \triangleright M & & Y \otimes X \triangleright M
 \end{array}$$

$$\begin{array}{ccc}
 X \otimes Y \triangleright M & \xrightarrow{e_{X, Y \triangleright M}} & X \otimes Y \triangleright M \\
 \downarrow c_{X, Y} \triangleright \text{id}_M & \searrow & \downarrow c_{Y, X} \triangleright \text{id}_M \\
 Y \otimes X \triangleright M & \xrightarrow{\text{id}_Y \otimes e_{X, M}} & Y \otimes X \triangleright M
 \end{array}$$

BRAIDED $\mathcal{C}$ -MODULE CATEGORY

$(\mathcal{M}, \triangleright, e)$

CATEGORY BIFUNCTOR $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$

$$\begin{array}{c} M \\ | \end{array} \quad \begin{array}{c} X \triangleright M \\ | \end{array} = \begin{array}{c} X \\ | \end{array} \begin{array}{c} M \\ | \end{array}$$

$e := \{ e_{X, M} : X \triangleright M \xrightarrow{\sim} X \triangleright M \}_{X \in \mathcal{C}, M \in \mathcal{M}}$
 NAT. ISOM.
 SUCH THAT

$$\begin{array}{|c|} \hline \text{Braid Axiom 1} \\ \hline \end{array}
 \quad
 \begin{array}{|c|} \hline \text{Braid Axiom 2} \\ \hline \end{array}$$

$$\begin{array}{c} X \\ \curvearrowright \end{array} \begin{array}{c} M \\ | \end{array}$$

"BRAID AXIOMS"

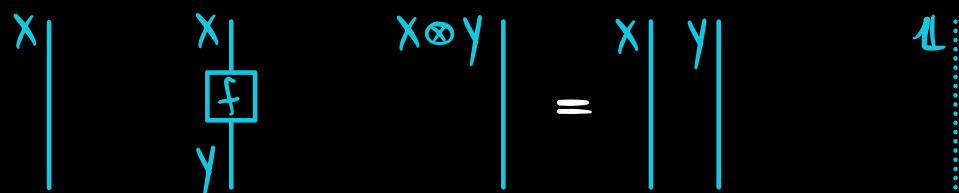
JOYAL-STREET
(1986, 1993)

BRAIDED-TYPE CATEGORIES

BROCHIER (2013)
FIX ($\mathcal{C}, \otimes, \mathbb{1}, c$)

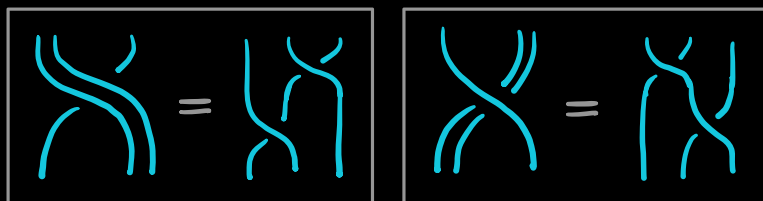
BRAIDED MONOIDAL CATEGORY ($\mathcal{C}, \otimes, \mathbb{1}, c$)

CATEGORY $f: x \rightarrow y$ BIFUNCTOR $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ DIST. OBJECT IN $\mathcal{C}$



$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
NAT. ISOM. SUCH THAT

"BRAID AXIOMS"

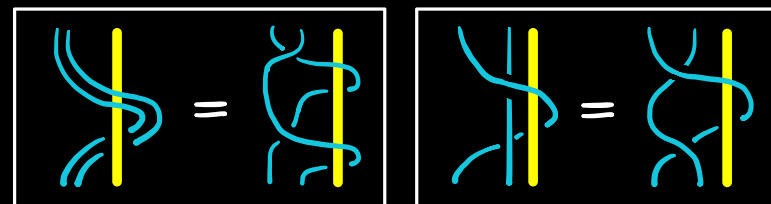


BRAIDED $\mathcal{C}$ -MODULE CATEGORY ($\mathcal{M}, \triangleright, e$)

CATEGORY BIFUNCTOR $\mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$



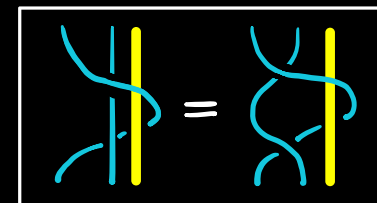
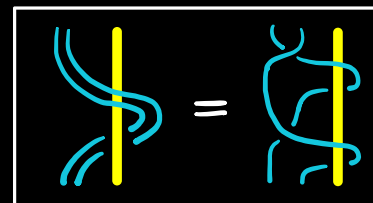
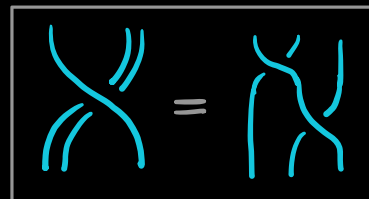
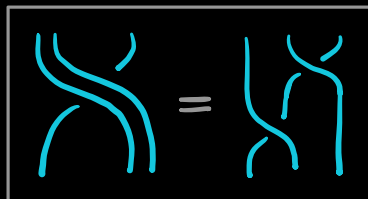
$e := \{e_{x,M} : x \triangleright M \xrightarrow{\sim} x \triangleright M\}_{x \in \mathcal{C}, M \in \mathcal{M}}$
NAT. ISOM. SUCH THAT



"BRAID AXIOMS"

TIES TO BRAID GROUP REP'NS

"BRAID
AXIOMS"



"BRAID
AXIOMS"

$\equiv \text{Type A} \equiv$

TIES TO BRAID GROUP REP'NS



COXETER GRAPH
OF TYPE A_n

$\equiv \text{TYPE A} \equiv$

TIES TO BRAID GROUP REP'NS



COXETER GRAPH
OF TYPE A_n

B_n^A

BRAID GROUP OF TYPE A_n

GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS:
$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{FOR } |i-j| \geq 2 \end{cases}$$

$\equiv \text{TYPE A} \equiv$

TIES TO BRAID GROUP REP'NS



COXETER GRAPH
OF TYPE A_n

$B\Gamma_n^A$ BRAID GROUP OF TYPE A_n

GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS:
$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{cases}$$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

$$\rho_n^X : B\Gamma_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$\equiv \text{TYPE A} \equiv$

TIES TO BRAID GROUP REP'NS



COXETER GRAPH
OF TYPE A_n

$B\Gamma_n^A$ BRAID GROUP OF TYPE A_n

GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS:
$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{cases}$$

FOR $\chi \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

$$\rho_n^\chi : B\Gamma_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(\chi^{\otimes n})$$

$$\sigma_i \longmapsto \text{id}_{\chi^{\otimes i-1}} \otimes c_{\chi, \chi} \otimes \text{id}_{\chi^{\otimes n-i-1}}$$

$\equiv \text{TYPE A} \equiv$

TIES TO BRAID GROUP REP'NS



COXETER GRAPH
OF TYPE A_n

B_n^A BRAID GROUP OF TYPE A_n

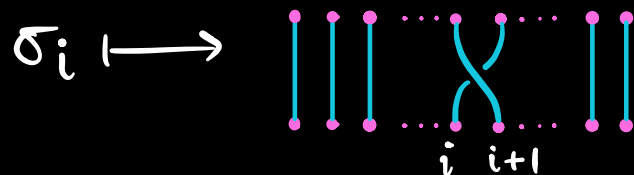
GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS:
$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{cases}$$

FOR $\chi \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

$$\rho_n^\chi : B_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(\chi^{\otimes n})$$



$\equiv \text{TYPE A} \equiv$

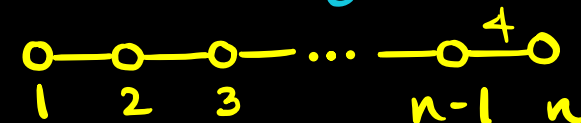
TIES TO BRAID GROUP REP'NS

$\equiv \text{TYPE B}^{\text{C}} \equiv$



COXETER GRAPH
OF TYPE A_n

COXETER GRAPH
OF TYPE B_n



B_n^A BRAID GROUP OF TYPE A_n

GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS:
$$\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{cases}$$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

$$\rho_n^X : B_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

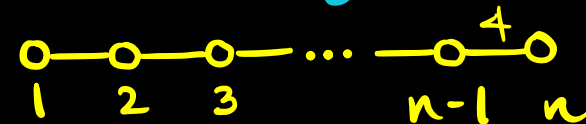
$$\sigma_i \longmapsto \begin{array}{c} \text{Diagram showing a crossing of strands } i \text{ and } i+1 \end{array}$$

≡ TYPE A ≡

TIES TO BRAID GROUP REPR'S

$$\equiv \text{TYPE } B \stackrel{C}{=} \equiv$$


COXETER GRAPH OF TYPE A_n



COXETER GRAPH OF TYPE B_n

$$B_n^A \quad \text{BRAID GROUP OF TYPE } A_n$$

GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS:
$$\left\{ \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right.$$

B_n^B BRAID GROUP OF TYPE B_n

GENERATORS: $\sigma_1, \dots, \sigma_{n-1}, t$

RELATIONS: $\left\{ \begin{array}{l} \text{TYPE A RELATIONS FOR } \sigma_i \\ \sigma_i t = t \sigma_i \text{ FOR } i=1, \dots, n-2 \\ \sigma_{n-1} t \sigma_{n-1} t = t \sigma_{n-1} t \sigma_{n-1} \end{array} \right.$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

$$\rho_n^X : Br_{n-1}^A \longrightarrow \text{Aut}_e(X^{\otimes n})$$

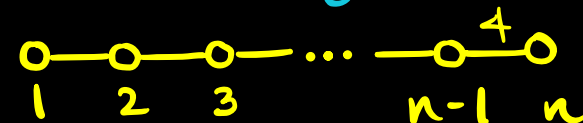
$$\sigma_i \longmapsto \begin{array}{ccccccc} | & | & | & \cdots & & \cdots & | & | \\ & & & & \searrow & \nearrow & & \\ & & & & i & i+1 & & \\ & & & & \nearrow & \searrow & & \\ & & & & & & & \\ | & | & | & \cdots & & \cdots & | & | \end{array}$$

≡ TYPE A ≡

TIES TO BRAID GROUP REP'NS

$$\equiv \text{TYPE } B \stackrel{C}{=} \text{C}$$


COXETER GRAPH OF TYPE A_n



COXETER GRAPH OF TYPE B_n

$$B_n^A \quad \text{BRAID GROUP OF TYPE } A_n$$

GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS:
$$\left\{ \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right.$$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

$$\rho_n^X : Br_{n-1}^A \longrightarrow \text{Aut}_e(X^{\otimes n})$$

$$\sigma_i \longmapsto \begin{array}{c} \text{Diagram with } i \text{ vertical lines, a crossing between lines } i \text{ and } i+1, \text{ and } i+1 \text{ vertical lines.} \end{array}$$

B_n^B BRAID GROUP OF TYPE B_n

GENERATORS: $\sigma_1, \dots, \sigma_{n-1}, t$

RELATIONS: $\left\{ \begin{array}{l} \text{TYPE A RELATIONS FOR } \sigma_i \\ \sigma_i t = t \sigma_i \text{ FOR } i=1, \dots, n-2 \\ \sigma_{n-1} t \sigma_{n-1} t = t \sigma_{n-1} t \sigma_{n-1} \end{array} \right.$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ & $M \in (\mathcal{M}, \triangleright, e)$ $BR \text{ MOD CAT.}$

GET
GROUP
HOMOM:

$$\rho_n^{X,M}: \text{Br}_n^B \longrightarrow \text{Aut}_m(X^{\otimes n} \triangleright M)$$

$\equiv \text{TYPE A} \equiv$

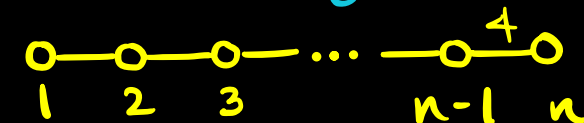
TIES TO BRAID GROUP REP'NS

$\equiv \text{TYPE B}^{\mathbb{C}} \equiv$



COXETER GRAPH
OF TYPE A_n

COXETER GRAPH
OF TYPE B_n



B_n^A BRAID GROUP OF TYPE A_n

GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS: $\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{cases}$

B_n^B BRAID GROUP OF TYPE B_n

GENERATORS: $\sigma_1, \dots, \sigma_{n-1}, t$

RELATIONS: $\begin{cases} \text{TYPE A RELATIONS FOR } \sigma_i \\ \sigma_i t = t \sigma_i \text{ FOR } i=1, \dots, n-2 \\ \sigma_{n-1} t \sigma_{n-1} t = t \sigma_{n-1} t \sigma_{n-1} \end{cases}$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

$$\rho_n^X: B_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \longmapsto \begin{array}{c} \text{Diagram of a braid with } n \text{ strands, where strands } i \text{ and } i+1 \text{ cross.} \end{array}$$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c) \nsubseteq M \in (\mathcal{M}, \triangleright, e)$ BR MOD CAT.

$$\text{GET GROUP HOMOM: } \rho_n^{X,M}: B_n^B \longrightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$$

$$\sigma_i \longmapsto \text{id}_{X^{\otimes i-1}} \otimes c_{X,X} \otimes \text{id}_{X^{\otimes n-i-1}}$$

$$t \longmapsto \text{id}_{X^{\otimes n-1}} \otimes e_{X,M}$$

$\equiv \text{TYPE A} \equiv$

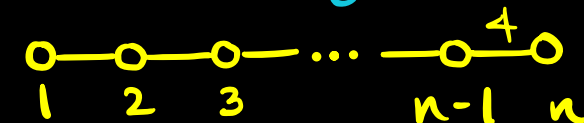
TIES TO BRAID GROUP REP'NS

$\equiv \text{TYPE B}^{\mathbb{C}} \equiv$



COXETER GRAPH
OF TYPE A_n

COXETER GRAPH
OF TYPE B_n



B_n^A BRAID GROUP OF TYPE A_n

GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS: $\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{cases}$

B_n^B BRAID GROUP OF TYPE B_n

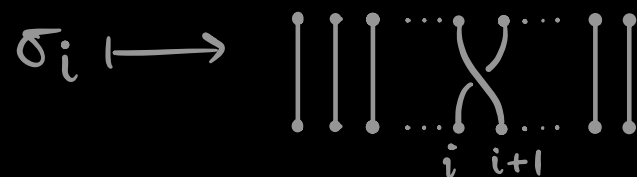
GENERATORS: $\sigma_1, \dots, \sigma_{n-1}, t$

RELATIONS: $\begin{cases} \text{TYPE A RELATIONS FOR } \sigma_i \\ \sigma_i t = t \sigma_i \text{ FOR } i=1, \dots, n-2 \\ \sigma_{n-1} t \sigma_{n-1} t = t \sigma_{n-1} t \sigma_{n-1} \end{cases}$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

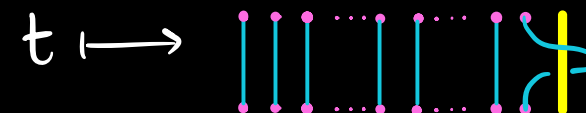
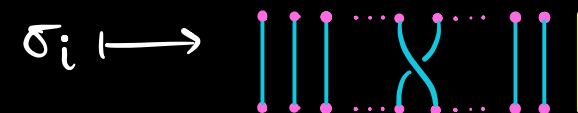
$$\rho_n^X: B_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$



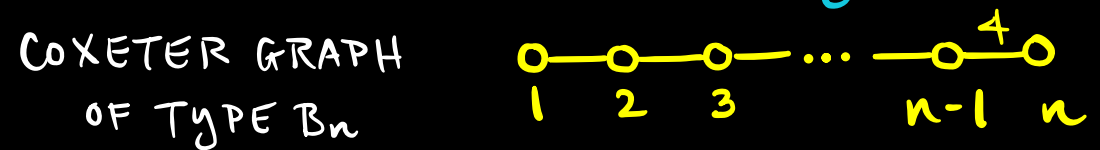
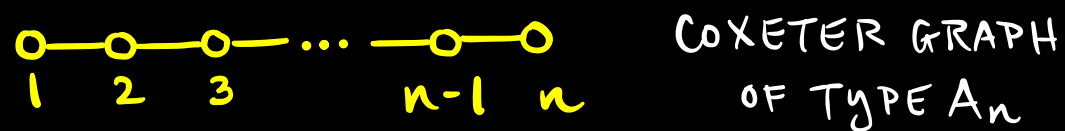
FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c) \nmid M \in (\mathcal{M}, \triangleright, e)$ BR MOD CAT.

GET GROUP HOMOM: $\rho_n^{X,M}: B_n^B \longrightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$

HOMOM:



$\equiv \text{TYPE A} \equiv$ TIES TO BRAID GROUP REP'NS $\equiv \text{TYPE B} \stackrel{C}{\equiv}$



$B\Gamma_n^A$ BRAID GROUP OF TYPE A_n

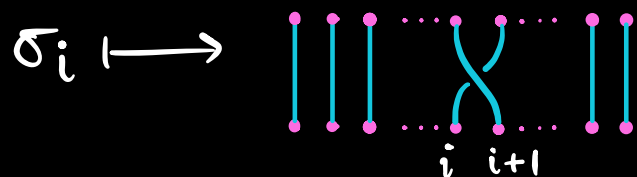
GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS: $\begin{cases} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \text{FOR } i=1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{cases}$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

$$\rho_n^X: B\Gamma_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$



$B\Gamma_n^B$ BRAID GROUP OF TYPE B_n

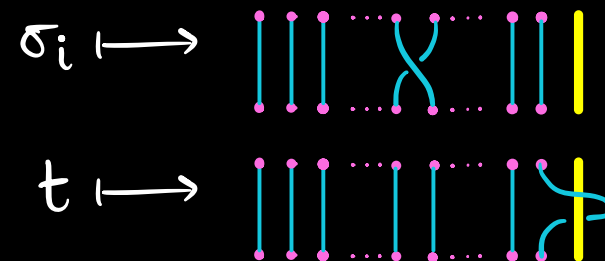
GENERATORS: $\sigma_1, \dots, \sigma_{n-1}, t$

RELATIONS: $\begin{cases} \text{TYPE A RELATIONS FOR } \sigma_i \\ \sigma_i t = t \sigma_i \text{ FOR } i=1, \dots, n-2 \\ \sigma_{n-1} t \sigma_{n-1} t = t \sigma_{n-1} t \sigma_{n-1} \end{cases}$

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c) \nmid M \in (\mathcal{M}, \triangleright, e)$ BR MOD CAT.

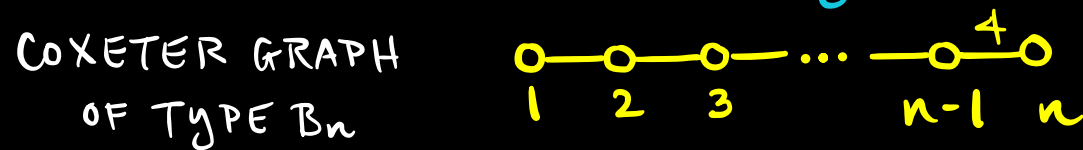
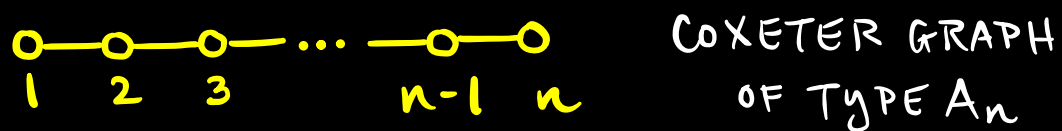
GET GROUP HOMOM: $\rho_n^{X,M}: B\Gamma_n^B \longrightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$

HOMOM:



TIES TO BRAID GROUP REP'NS

$\equiv \text{TYPE A} \equiv$
 $\equiv \text{TYPE B} \stackrel{C}{\equiv}$



B_n^A BRAID GROUP OF TYPE A_n

GENERATORS: $\sigma_1, \dots, \sigma_n$

RELATIONS: $\left\{ \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right.$

qYBE

B_n^B BRAID GROUP OF TYPE B_n

GENERATORS: $\sigma_1, \dots, \sigma_{n-1}, t$

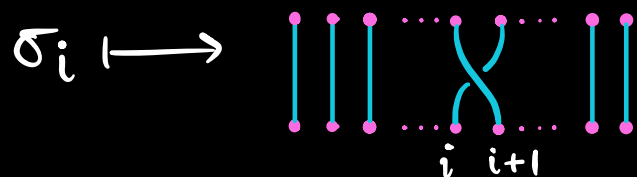
RELATIONS: $\left\{ \begin{array}{l} \text{TYPE A RELATIONS FOR } \sigma_i \\ \sigma_i t = t \sigma_i \text{ FOR } i=1, \dots, n-2 \\ \sigma_{n-1} t \sigma_{n-1} t = t \sigma_{n-1} t \sigma_{n-1} \end{array} \right.$

qRE

FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ BR. MONOIDAL CAT.

GET GROUP HOMOM:

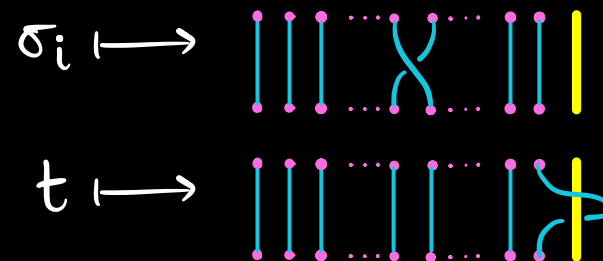
$$\rho_n^X: B_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$



FOR $X \in (\mathcal{C}, \otimes, \mathbb{1}, c) \nmid M \in (\mathcal{M}, \triangleright, e)$ BR MOD CAT.

GET GROUP HOMOM: $\rho_n^{X,M}: B_n^B \longrightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$

HOMOM:



$\equiv \text{Type A} \equiv$

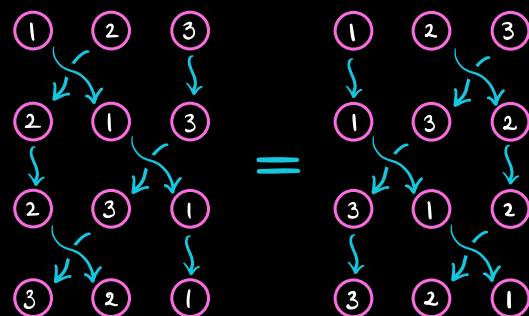
SUMMARY ... SO FAR

$\equiv \text{Type B} \stackrel{C}{=} \equiv$

≡ TYPE A ≡

SUMMARY ... SO FAR

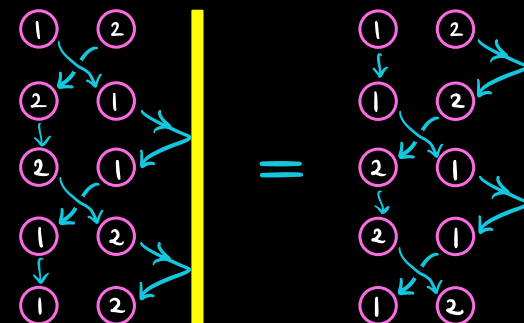
≡ TYPE B^C ≡



$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

≡ WANT SOLUTIONS ≡



$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

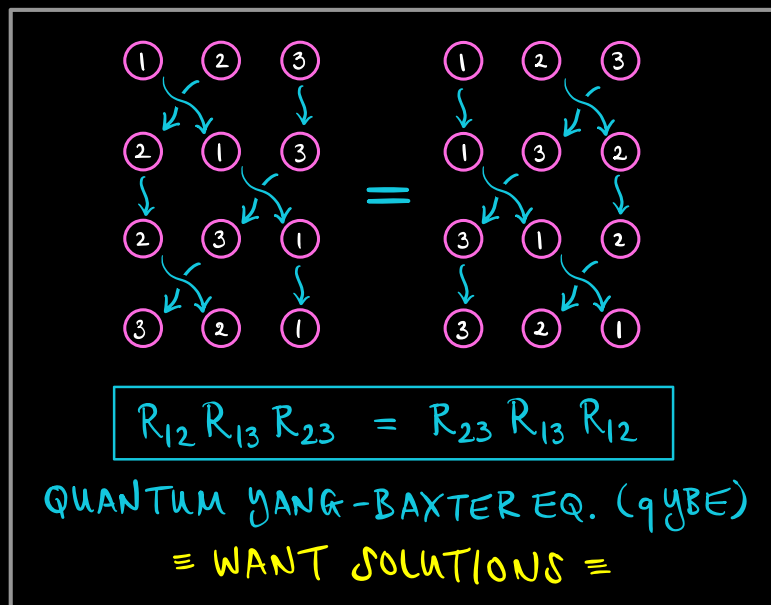
QUANTUM REFLECTION EQ (qRE)

≡ WANT SOLUTIONS ≡

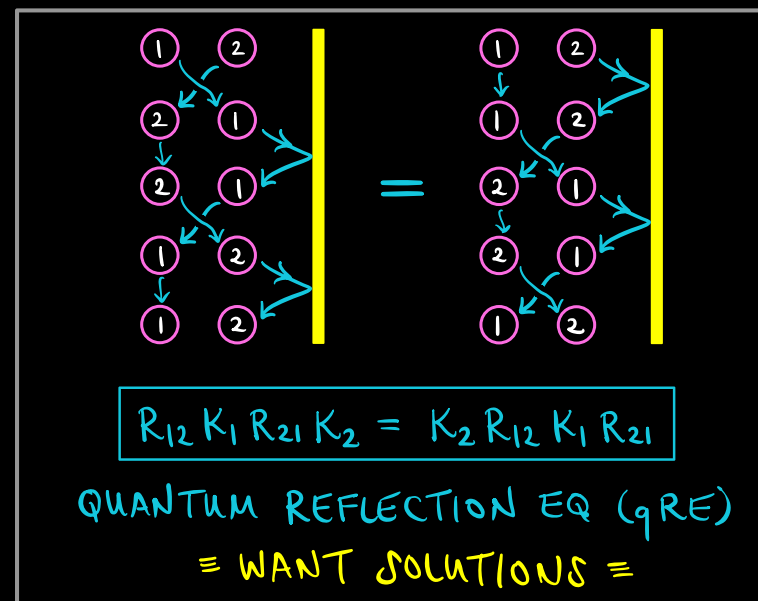
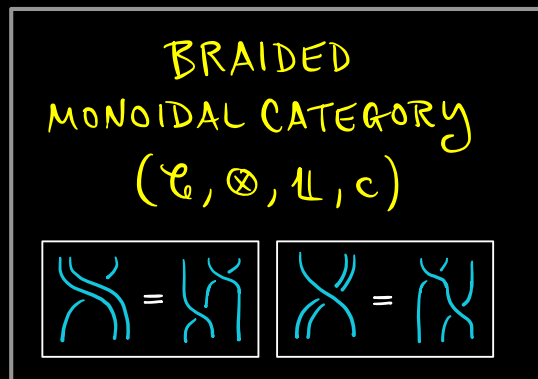
$\equiv \text{Type A} \equiv$

SUMMARY ... SO FAR

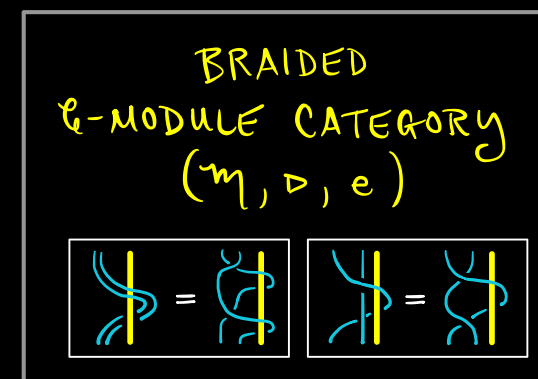
$\equiv \text{Type B}^{\mathbb{C}} \equiv$



OBJECTS



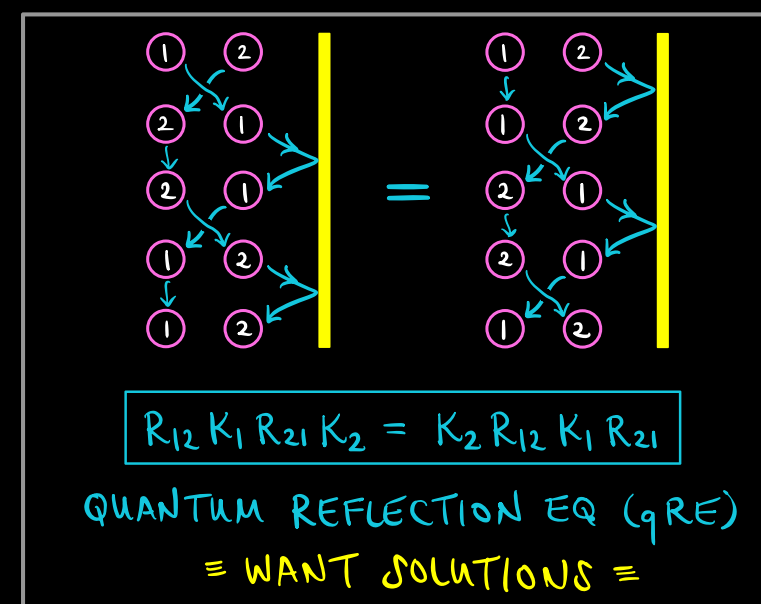
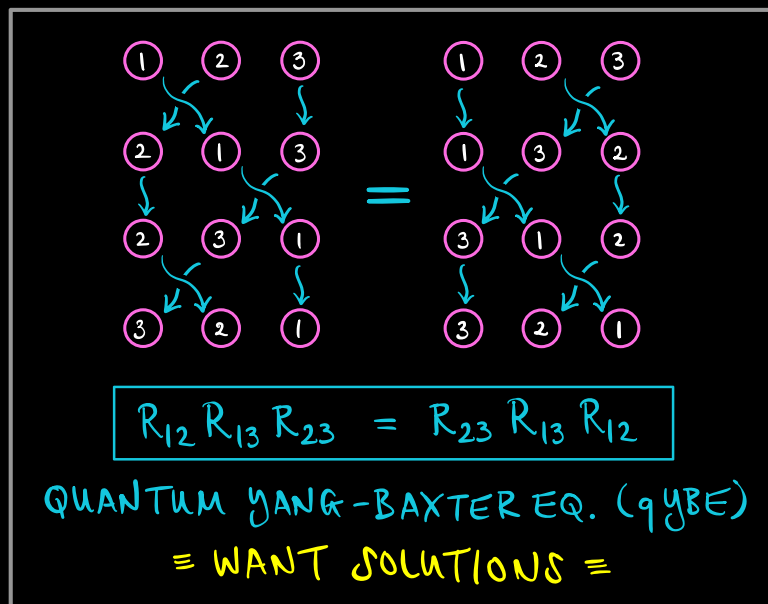
OBJECTS



$\equiv \text{TYPE A} \equiv$

SUMMARY ... SO FAR

$\equiv \text{TYPE B} \stackrel{C}{\equiv}$



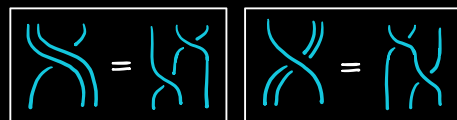
OBJECTS

OBJECTS

OBJECTS

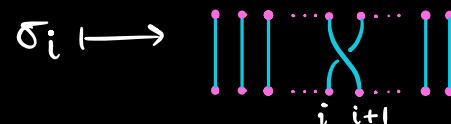
OBJECTS

BRAIDED
MONOIDAL CATEGORY
($\mathcal{C}, \otimes, \mathbb{1}, c$)



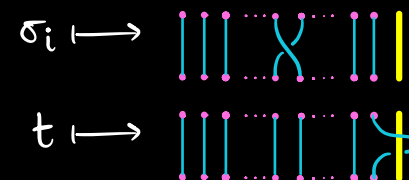
BRAID GROUP OF TYPE A

$$\rho_n^X: Br_{n-1}^A \rightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

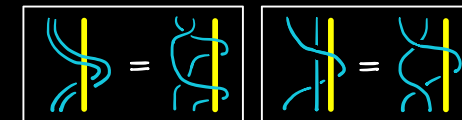


BRAID GROUP OF TYPE B

$$\rho_n^{X,M}: Br_n^B \rightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$$



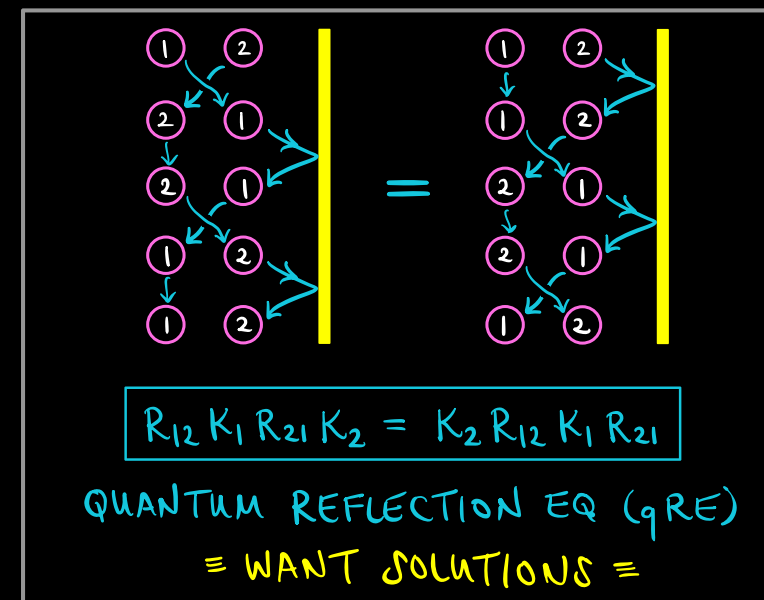
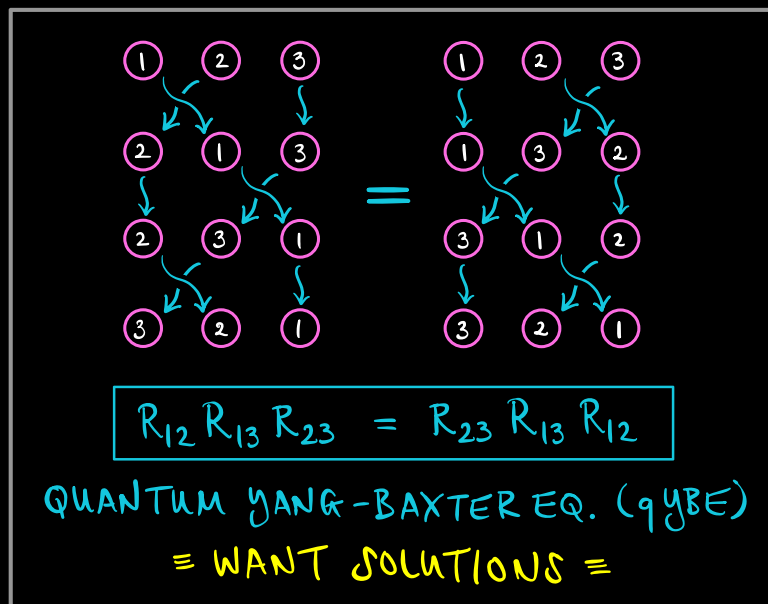
BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
($\mathcal{M}, \triangleright, e$)



$\equiv \text{Type A} \equiv$

SUMMARY ... SO FAR

$\equiv \text{Type B} \stackrel{C}{\equiv}$



OBJECTS

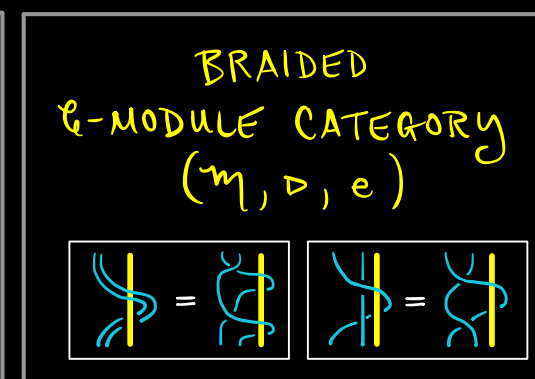
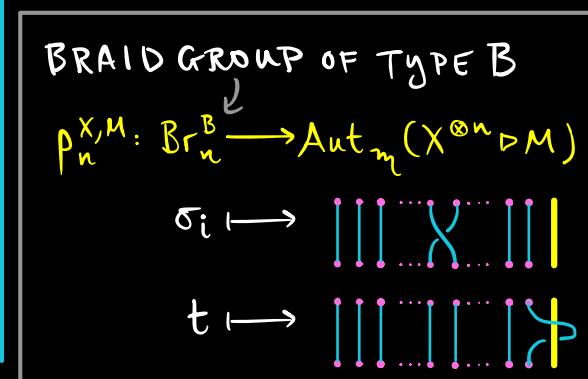
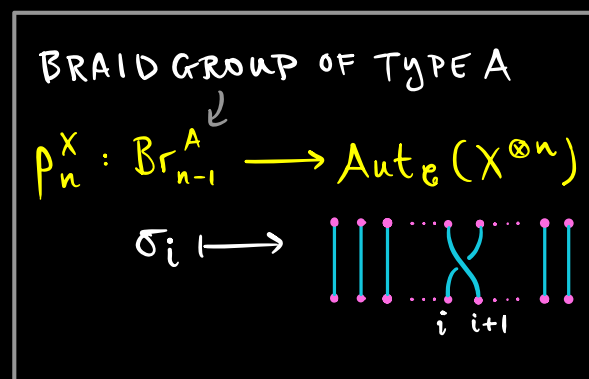
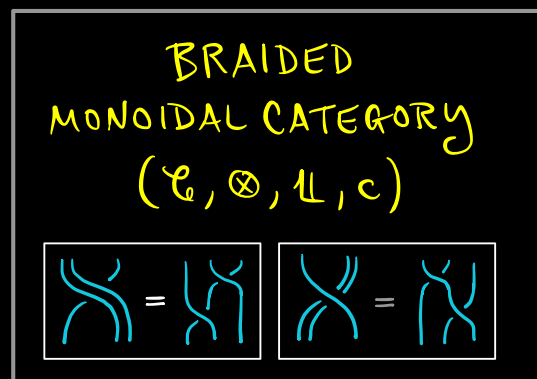
OBJECTS

FOR GENERAL
REPS OF Br^A

FOR GENERAL
REPS OF Br^B

OBJECTS

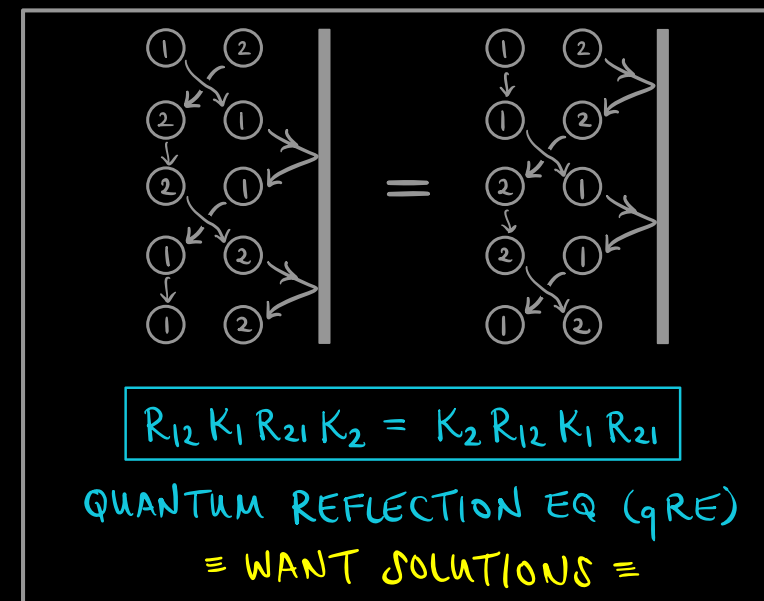
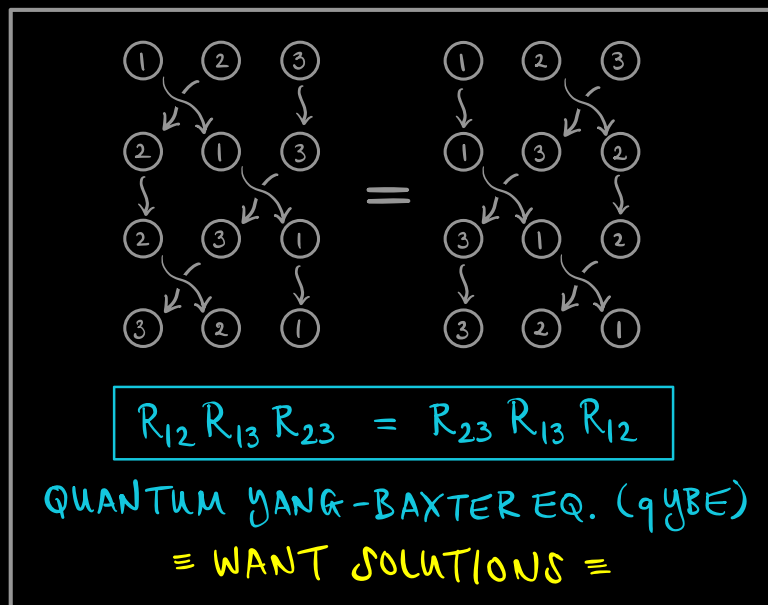
OBJECTS



$\equiv \text{Type A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{Type B} \stackrel{C}{\equiv}$



OBJECTS

OBJECTS

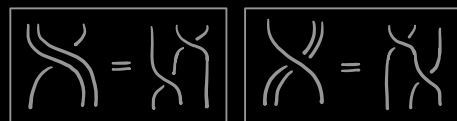
FOR GENERAL
REP'NS OF Br^A

FOR GENERAL
REP'NS OF Br^B

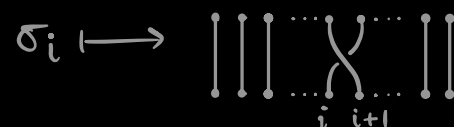
OBJECTS

OBJECTS

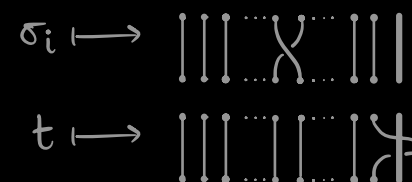
BRAIDED
MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$



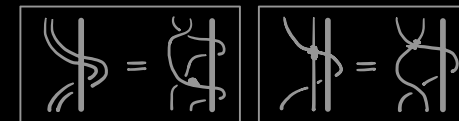
BRAID GROUP OF TYPE A
 $\rho_n^X: Br_{n-1}^A \rightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$



BRAID GROUP OF TYPE B
 $\rho_n^{X,M}: Br_n^B \rightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$



BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



$\equiv \text{TYPE A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{TYPE B}^{\text{C}} \equiv$

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

$\equiv \text{WANT SOLUTIONS} \equiv$

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

$\equiv \text{WANT SOLUTIONS} \equiv$

$\equiv \text{Type A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{Type B}^{\text{C}} \equiv$

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)

$\mathcal{C} := H\text{-Mod}$

MONOIDAL CATEGORY

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

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$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

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$\equiv \text{WANT SOLUTIONS} \equiv$

$\equiv \text{Type A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{Type B}^{\text{C}} \equiv$

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)

INV. ELT. OF $H \otimes H$ + AXIOMS $\uparrow$

$\mathcal{C} := H\text{-Mod}$

MONOIDAL CATEGORY

"R-MATRIX" R

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

$\equiv \text{WANT SOLUTIONS} \equiv$

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

$\equiv \text{WANT SOLUTIONS} \equiv$

≡ TYPE A ≡

HOPF-TYPE SOLUTIONS

≡ TYPE B^C ≡

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)

INV. ELT. OF $H \otimes H$ + AXIOMS 

$\mathcal{C} := H\text{-Mod}$ BRAIDED MONOIDAL CATEGORY

BRAIDING $C \longleftrightarrow$ "R-MATRIX" R

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

≡ WANT SOLUTIONS ≡  

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

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≡ WANT SOLUTIONS ≡

$\equiv \text{TYPE A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{TYPE B}^{\text{C}} \equiv$

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QUANTUM YANG-BAXTER EQ. (qYBE)

$\equiv \text{WANT SOLUTIONS} \equiv \checkmark$

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

$\equiv \text{WANT SOLUTIONS} \equiv$

ON "SAMENESS" OF SOLUTIONS OF qYBE

$=$ BRAIDED MORITA EQUIVALENCE:

$$(H, R)\text{-Mod} \simeq (H', R')\text{-Mod}$$

AS BRAIDED MONOIDAL CATEGORIES

$\equiv \text{TYPE A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{TYPE B}^{\text{C}} \equiv$

THEOREM [SHIMIZU, 2010] TAKE F.D.
QUASITRIANGULAR HOPF ALGAS. $(H, R) \ncong (H', R')$.

$(H, R) \ncong (H', R')$ BR. MORITA EQUIV.

$\Rightarrow \rho_n^H \simeq \rho_n^{H'}$ AS REP'NS OF B_{n-1}^A

$$[\rho_n^H: B_{n-1}^A \rightarrow \text{Aut}_{H\text{-Mod}}(H^{\otimes n})]$$

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv$ WANT SOLUTIONS $\equiv$

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HOPF-TYPE SOLUTIONS

≡ TYPE B^C ≡

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QUASID HOPF ALGS. $(H, R) \nless (H', R')$.

$(H, R) \nless (H', R')$ BR. MORITA EQUIV.

$\Rightarrow \rho_n^H \simeq \rho_n^{H'}$ AS REP'NS OF B_{n-1}^A

EXAMPLE TAKE FINITE GROUPS G, G' .

$\leadsto (kG, R=e \otimes e)$ IS A QUASID HOPF ALG.

NOW $\text{trace}(\rho_n^G(e_{B_{n-1}^A})) = |G|^n$

$\therefore (kG, e \otimes e) \simeq (kG', e' \otimes e')$ BR. MOR. EQ

$\Rightarrow |G| = |G'|$

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
≡ WANT SOLUTIONS ≡

$\equiv \text{TYPE A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{TYPE B}^{\mathbb{C}} \equiv$

THEOREM [SHIMIZU, 2010] TAKE F.D.
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$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

$\equiv \text{WANT SOLUTIONS} \equiv$

≡ TYPE A ≡

HOPF-TYPE SOLUTIONS

≡ TYPE B^C ≡

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)

INV. ELT. OF $H \otimes H$ + AXIOMS 

$\mathcal{C} := H\text{-Mod}$ BRAIDED MONOIDAL CATEGORY

BRAIDING $C \longleftrightarrow$ "R-MATRIX" R

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

≡ WANT SOLUTIONS ≡  

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

≡ WANT SOLUTIONS ≡

ON "SAMENESS" OF SOLUTIONS OF qYBE

= BRAIDED MORITA EQUIVALENCE:

$$(H, R)\text{-Mod} \simeq (H', R')\text{-Mod}$$

AS BRAIDED MONOIDAL CATEGORIES

$\equiv \text{TYPE A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{TYPE B} \stackrel{C}{\equiv}$

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)
INV. ELT. OF $H \otimes H$ + AXIOMS $\uparrow$

$\mathcal{C} := H\text{-Mod}$ BRAIDED MONOIDAL CATEGORY

BRAIDING $C \longleftrightarrow$ "R-MATRIX" R

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv$ WANT SOLUTIONS $\equiv \checkmark$

FIX A QUASIA HOPF ALG. (H, R)

$\nmid$ TAKE A QUASIA H -COMODULE ALG. (A, K)

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv$ WANT SOLUTIONS $\equiv$

ON "SAMENESS" OF SOLUTIONS OF qYBE

$=$ BRAIDED MORITA EQUIVALENCE:

$$(H, R)\text{-Mod} \simeq (H', R')\text{-Mod}$$

AS BRAIDED MONOIDAL CATEGORIES

$\equiv \text{TYPE A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{TYPE B}^{\mathbb{C}} \equiv$

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)
INV. ELT. OF $H \otimes H$ + AXIOMS $\uparrow$

$\mathcal{C} := H\text{-Mod}$ BRAIDED MONOIDAL CATEGORY

BRAIDING $C \longleftrightarrow$ "R-MATRIX" R

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv$ WANT SOLUTIONS $\equiv \checkmark$

FIX A QUASIA HOPF ALG. (H, R)

$\nmid$ TAKE A QUASIA H -COMODULE ALG. (A, K)

$\mathcal{M} := A\text{-Mod}$

$\mathcal{C}$ -MODULE CATEGORY

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
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$\equiv \text{TYPE A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{TYPE B} \stackrel{C}{\equiv}$

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)
INV. ELT. OF $H \otimes H$ + AXIOMS $\uparrow$

$\mathcal{C} := H\text{-Mod}$ BRAIDED MONOIDAL CATEGORY

BRAIDING $C \longleftrightarrow$ "R-MATRIX" R

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv$ WANT SOLUTIONS $\equiv \checkmark$

FIX A QUASIA HOPF ALG. (H, R)

$\nmid$ TAKE A QUASIA H -COMODULE ALG. (A, K)
[KOLB, 2020] INV. ELT. OF $H \otimes A$ + AXIOMS $\uparrow$

$\mathcal{M} := A\text{-Mod}$

$\mathcal{C}$ -MODULE CATEGORY

"K-MATRIX" K

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv$ WANT SOLUTIONS $\equiv$

ON "SAMENESS" OF SOLUTIONS OF qYBE

$=$ BRAIDED MORITA EQUIVALENCE:

$$(H, R)\text{-Mod} \simeq (H', R')\text{-Mod}$$

AS BRAIDED MONOIDAL CATEGORIES

$\equiv \text{TYPE A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{TYPE B} \stackrel{C}{\equiv}$

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)
INV. ELT. OF $H \otimes H$ + AXIOMS $\uparrow$

$\mathcal{C} := H\text{-Mod}$ BRAIDED MONOIDAL CATEGORY

BRAIDING $c \longleftrightarrow$ "R-MATRIX" R

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv$ WANT SOLUTIONS $\equiv \checkmark$

FIX A QUASIA HOPF ALG. (H, R)

$\nmid$ TAKE A QUASIA H -COMODULE ALG. (A, K)
[KOLB, 2020] INV. ELT. OF $H \otimes A$ + AXIOMS $\uparrow$

$\mathcal{M} := A\text{-Mod}$ BRAIDED $\mathcal{C}$ -MODULE CATEGORY

BRAIDING $e \longleftrightarrow$ "K-MATRIX" K

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv$ WANT SOLUTIONS $\equiv \checkmark$

ON "SAMENESS" OF SOLUTIONS OF qYBE

$=$ BRAIDED MORITA EQUIVALENCE:

$$(H, R)\text{-Mod} \simeq (H', R')\text{-Mod}$$

AS BRAIDED MONOIDAL CATEGORIES

$\equiv \text{TYPE A} \equiv$

HOPF-TYPE SOLUTIONS

$\equiv \text{TYPE B} \stackrel{C}{\equiv}$

TAKE A QUASITRIANGULAR HOPF ALG. (H, R)
INV. ELT. OF $H \otimes H$ + AXIOMS $\uparrow$

$\mathcal{C} := H\text{-Mod}$ BRAIDED MONOIDAL CATEGORY

BRAIDING $c \longleftrightarrow$ "R-MATRIX" R

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv$ WANT SOLUTIONS $\equiv \checkmark$

FIX A QUASIA HOPF ALG. (H, R)

$\nmid$ TAKE A QUASIA H -COMODULE ALG. (A, K)
[KOLB, 2020] INV. ELT. OF $H \otimes A$ + AXIOMS $\uparrow$

$\mathcal{M} := A\text{-Mod}$ BRAIDED $\mathcal{C}$ -MODULE CATEGORY

BRAIDING $e \longleftrightarrow$ "K-MATRIX" K

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv$ WANT SOLUTIONS $\equiv \checkmark$

ON "SAMENESS" OF SOLUTIONS OF qYBE

$=$ BRAIDED MORITA EQUIVALENCE:

$$(H, R)\text{-Mod} \simeq (H', R')\text{-Mod}$$

AS BRAIDED MONOIDAL CATEGORIES

ON "SAMENESS" OF SOLUTIONS OF qRE

$=$ BRAIDED MORITA EQUIVALENCE:

$$(A, K)\text{-Mod} \simeq (A', K')\text{-Mod}$$

AS BRAIDED $\mathcal{C}$ -MODULE CATEGORIES

≡ TYPE A ≡

HOPF-TYPE SOLUTIONS

≡ TYPE B^C ≡

THEOREM [SHIMIZU, 2010] TAKE F.D.
QUASIA HOPF ALGS. (H, R) & (H', R') .

(H, R) & (H', R') BR. MORITA EQUIV.

⇒ $\rho_n^H \simeq \rho_n^{H'}$ AS REP'NS OF B_{n-1}^A

$$[\rho_n^H: B_{n-1}^A \rightarrow \text{Aut}_{H\text{-Mod}}(H^{\otimes n})]$$

ON "SAMENESS" OF SOLUTIONS OF qYBE

= BRAIDED MORITA EQUIVALENCE:

$$(H, R)\text{-Mod} \simeq (H', R')\text{-Mod}$$

AS BRAIDED MONOIDAL CATEGORIES

FIX A QUASIA HOPF ALG. (H, R)

& TAKE A QUASIA H-COMODULE ALG. (A, K)

[KOLB, 2020] INV. ELT. OF $H \otimes A$ + AXIOMS

$\mathcal{M} := A\text{-Mod}$ BRAIDED $\mathcal{C}$ -MODULE CATEGORY

BRAIDING $e \longleftrightarrow$ "K-MATRIX" K

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

≡ WANT SOLUTIONS ≡ ✓

ON "SAMENESS" OF SOLUTIONS OF qRE

= BRAIDED MORITA EQUIVALENCE:

$$(A, K)\text{-Mod} \simeq (A', K')\text{-Mod}$$

AS BRAIDED $\mathcal{C}$ -MODULE CATEGORIES

≡ TYPE A ≡

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THEOREM [SHIMIZU, 2010] TAKE F.D.
QUASIA HOPF ALGS. $(H, R) \ncong (H', R')$.

$(H, R) \ncong (H', R')$ BR. MORITA EQUIV.

$\Rightarrow \rho_n^H \simeq \rho_n^{H'}$ AS REP'NS OF $B\Gamma_{n-1}^A$

$$[\rho_n^H: B\Gamma_{n-1}^A \rightarrow \text{Aut}_{H\text{-Mod}}(H^{\otimes n})]$$

ON "SAMENESS" OF SOLUTIONS OF qYBE

= BRAIDED MORITA EQUIVALENCE:

$$(H, R)\text{-Mod} \simeq (H', R')\text{-Mod}$$

AS BRAIDED MONOIDAL CATEGORIES

FIX A QUASIA HOPF ALG. (H, R) COMOD ALGS

THEOREM [MÜLLER-W, 2026] $\swarrow$ "IN" H

TAKE F.D. QUASIA COIDEAL SUBALGS OF (H, R) :

$$(A, K) \ncong (A', K')$$

THAT ARE BRAIDED MORITA EQUIV.

$\Rightarrow \rho_n^{H,A} \simeq \rho_n^{H,A'}$ AS REP'NS OF $B\Gamma_n^B$

$$[\rho_n^{H,A}: B\Gamma_n^B \rightarrow \text{Aut}_{A\text{-Mod}}(H^{\otimes n} \otimes A)]$$

ON "SAMENESS" OF SOLUTIONS OF qRE

= BRAIDED MORITA EQUIVALENCE:

$$(A, K)\text{-Mod} \simeq (A', K')\text{-Mod}$$

AS BRAIDED $\mathcal{C}$ -MODULE CATEGORIES

≡ TYPE A ≡

HOPF-TYPE SOLUTIONS

≡ TYPE B^C ≡

THEOREM [SHIMIZU, 2010] TAKE F.D.
QUASIA HOPF ALGS. $(H, R) \nmid (H', R')$.

$(H, R) \nmid (H', R')$ BR. MORITA EQUIV.

$\Rightarrow \rho_n^H \simeq \rho_n^{H'}$ AS REP'NS OF $B\Gamma_{n-1}^A$

EXAMPLE TAKE FINITE GROUPS G, G' .

$\leadsto (kG, R=e \otimes e)$ IS A QUASIA HOPF ALG

NOW $\text{trace}(\rho_n^G(e_{B\Gamma_{n-1}^A})) = |G|^n$

$\therefore (kG, e \otimes e) \simeq (kG', e' \otimes e')$ BR. MOR. EQ

$\Rightarrow |G| = |G'|$

FIX A QUASIA HOPF ALG. (H, R) COMOD ALGS

THEOREM [MÜLLER-W, 2026] $\swarrow$ "IN" H

TAKE F.D. QUASIA COIDEAL SUBALGS OF (H, R) :

$(A, K) \nmid (A', K')$

THAT ARE BRAIDED MORITA EQUIV.

$\Rightarrow \rho_n^{H,A} \simeq \rho_n^{H,A'}$ AS REP'NS OF $B\Gamma_n^B$

$[\rho_n^{H,A} : B\Gamma_n^B \rightarrow \text{Aut}_{A\text{-mod}}(H^{\otimes n} \otimes A)]$

ON "SAMENESS" OF SOLUTIONS OF qRE

= BRAIDED MORITA EQUIVALENCE:

$(A, K)\text{-Mod} \simeq (A', K')\text{-Mod}$

AS BRAIDED $\mathcal{C}$ -MODULE CATEGORIES

≡ TYPE A ≡

HOPF-TYPE SOLUTIONS

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THEOREM [SHIMIZU, 2010] TAKE F.D.
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EXAMPLE TAKE FINITE GROUPS G, G' .

$\leadsto (kG, R=e \otimes e)$ IS A QUASIA HOPF ALG

NOW $\text{trace}(\rho_n^G(e_{B_{n-1}^A})) = |G|^n$

$\therefore (kG, e \otimes e) \simeq (kG', e' \otimes e')$ BR. MOR. EQ

$\Rightarrow |G| = |G'|$

FIX A QUASIA HOPF ALG. (H, R) COMOD ALGS

THEOREM [MÜLLER-W, 2026] $\swarrow$ "IN" H

TAKE F.D. QUASIA COIDEAL SUBALGS OF (H, R) :

$(A, K) \nmid (A', K')$

THAT ARE BRAIDED MORITA EQUIV.

$\Rightarrow \rho_n^{H,A} \simeq \rho_n^{H,A'}$ AS REP'NS OF B_n^B

EXAMPLE TAKE A FINITE GROUP G
WITH SUBGROUPS L, L' .

LIKEWISE...

$(kL, e \otimes e) \simeq (kL', e \otimes e)$ BR. MOR. EQ

$\Rightarrow |L| = |L'|$

$\equiv \text{Type A} \equiv$

SUMMARY ... SO FAR

$\equiv \text{Type B}^{\mathbb{C}} \equiv$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

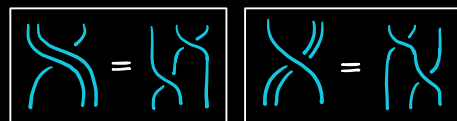
QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

OBJECTS

FOR GENERAL
 REPS OF Br^A

BRAIDED
 MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$



BRAID GROUP OF TYPE A

$$\rho_n^X: Br_{n-1}^A \rightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \mapsto \begin{array}{c} \text{diagram of } \sigma_i \text{ in } Br_n^A \end{array}$$

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

FOR GENERAL
 REPS OF Br^B

OBJECTS

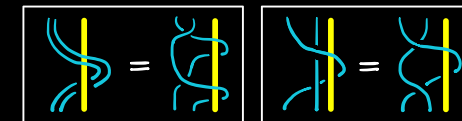
OBJECTS

BRAID GROUP OF TYPE B

$$\rho_n^{X,M}: Br_n^B \rightarrow \text{Aut}_M(X^{\otimes n} \triangleright M)$$

$$\begin{array}{l} \sigma_i \mapsto \text{diagram of } \sigma_i \text{ in } Br_n^B \\ t \mapsto \text{diagram of } t \text{ in } Br_n^B \end{array}$$

BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



$\equiv \text{TYPE A} \equiv$

SUMMARY ... SO FAR

$\equiv \text{TYPE B}^{\mathbb{C}} \equiv$

HOPF CASE:
QUASIA HOPF ALGS
... UP TO BR. MORITA EQUIV

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

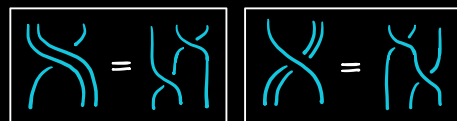
QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

OBJECTS

FOR GENERAL
REP'NS OF Br^A

BRAIDED
MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$



BRAID GROUP OF TYPE A

$$\rho_n^X: Br_{n-1}^A \rightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \mapsto \begin{array}{c} \text{diagram of a crossing between strands } i \text{ and } i+1 \end{array}$$

HOPF CASE:
QUASIA COMODULE ALGS
... UP TO BR. MORITA EQUIV

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

FOR GENERAL
REP'NS OF Br^B

OBJECTS

OBJECTS

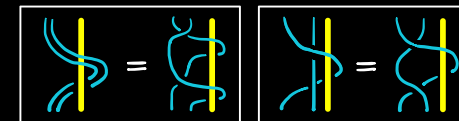
BRAID GROUP OF TYPE B

$$\rho_n^{X,M}: Br_n^B \rightarrow \text{Aut}_M(X^{\otimes n} \triangleright M)$$

$$\sigma_i \mapsto \begin{array}{c} \text{diagram of a crossing between strands } i \text{ and } i+1 \end{array}$$

$$t \mapsto \begin{array}{c} \text{diagram of a twist on a strand } i \end{array}$$

BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



$\equiv \text{Type A} \equiv$

SUMMARY ... SO FAR

$\equiv \text{Type B}^{\mathbb{C}} \equiv$

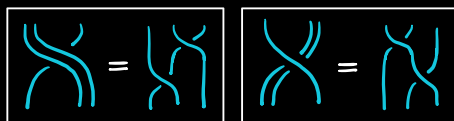
$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)

$\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

BRAIDED
MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, 1, c)$



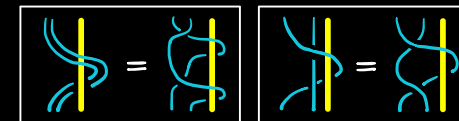
$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)

$\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

BRAIDED
 $\mathbb{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



$\equiv \text{Type A} \equiv$

CATEGORICAL CENTERS

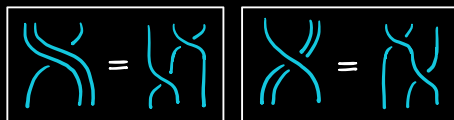
$\equiv \text{Type B}^{\mathbb{C}} \equiv$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

BRAIDED
MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, 1, c)$



LET'S
CONSTRUCT
THESE

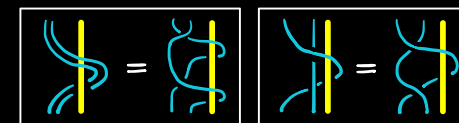


$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

BRAIDED
 $\mathbb{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



LET'S
CONSTRUCT
THESE



DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

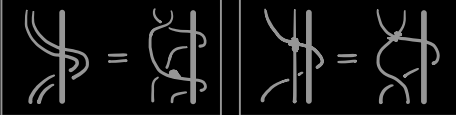
KASSEL
(1995)

CATEGORICAL CENTERS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$



$(\mathcal{M}, \triangleright, e)$



DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

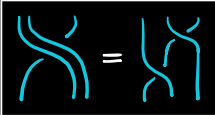
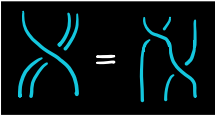
KASSEL
(1995)

CATEGORICAL CENTERS

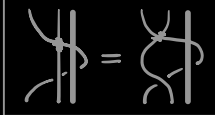
TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

 $=$ 

$(\mathcal{M}, \triangleright, e)$

 $=$ 

DRINFELD (1987)
 JOYAL-STREET (1991)
 MAJID (1991)

KASSEL
 (1995)

CATEGORICAL CENTERS

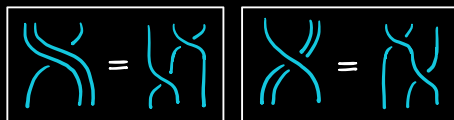
TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

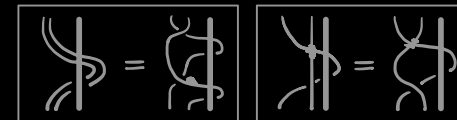
OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING...

$(\mathcal{C}, \otimes, \mathbb{1}, c)$



$(\mathcal{M}, \triangleright, e)$



DRINFELD (1987)
 JOYAL-STREET (1991)
 MAJID (1991)

KASSEL
 (1995)

CATEGORICAL CENTERS

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING $\text{Diagram 1} = \text{Diagram 2}$

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

$(\mathcal{M}, \triangleright, e)$

DRINFELD (1987)
 JOYAL-STREET (1991)
 MAJID (1991)

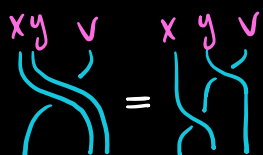
KASSEL
 (1995)

CATEGORICAL CENTERS

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

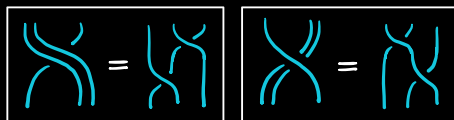
OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING 

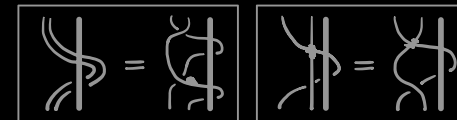
$\otimes : (V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V \otimes W})$

WITH $c_x^{V \otimes W} := \dots$

$(\mathcal{C}, \otimes, \mathbb{1}, c)$



$(\mathcal{M}, \triangleright, e)$



DRINFELD (1987)
 JOYAL-STREET (1991)
 MAJID (1991)

KASSEL
 (1995)

CATEGORICAL CENTERS

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

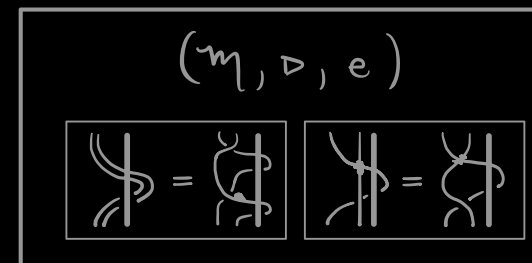
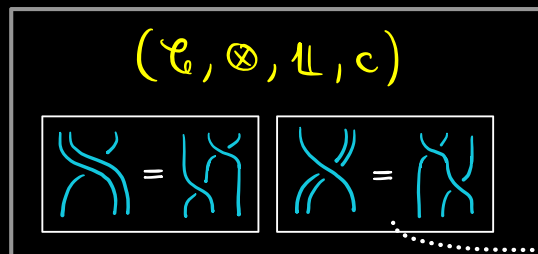
$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING $\begin{array}{c} x \quad y \quad V \\ \text{---} \quad \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} x \quad y \quad V \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array}$

$\otimes : (V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V \otimes W})$

WITH $c_x^{V \otimes W} := \begin{array}{c} x \quad V \quad W \\ \text{---} \quad \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} x \quad V \quad W \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array}$



DRINFELD (1987)
 JOYAL-STREET (1991)
 MAJID (1991)

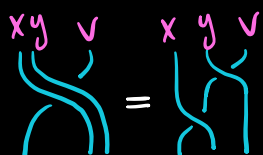
KASSEL
 (1995)

CATEGORICAL CENTERS

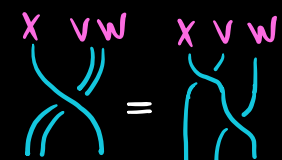
TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING 

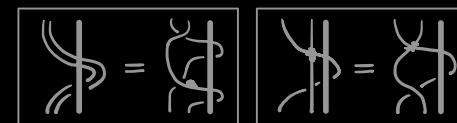
$\otimes : (V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V \otimes W})$

WITH $c_x^{V \otimes W} :=$ 

$c : (V, c^V) \otimes (W, c^W) \xrightarrow{c_V^W} (W, c^W) \otimes (V, c^V)$

$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

$(\eta, \triangleright, e)$



DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

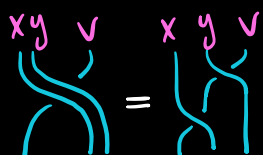
CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

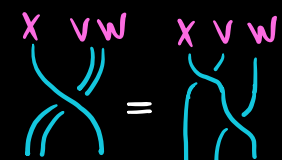
TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING 

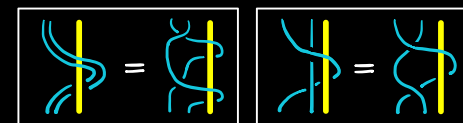
$\otimes : (V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V \otimes W})$

WITH $c_x^{V \otimes W} :=$ 

$c : (V, c^V) \otimes (W, c^W) \xrightarrow{c_V^W} (W, c^W) \otimes (V, c^V)$

$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

$(\eta, \triangleright, e)$



DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING $\begin{array}{c} x \quad y \quad V \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} x \quad y \quad V \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}$

$\otimes : (V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V \otimes W})$

WITH $c_x^{V \otimes W} := \begin{array}{c} x \quad V \quad W \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} x \quad V \quad W \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array}$

$c : (V, c^V) \otimes (W, c^W) \xrightarrow{c_V^W} (W, c^W) \otimes (V, c^V)$

$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

$(\mathcal{M}, \triangleright, e)$

$$\begin{array}{|c|} \hline \begin{array}{c} \text{blue line} \quad \text{yellow line} \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} \text{blue line} \quad \text{yellow line} \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \end{array} \quad \begin{array}{|c|} \hline \begin{array}{c} \text{blue line} \quad \text{yellow line} \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \text{blue line} \quad \text{yellow line} \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \end{array}$$

DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING $\text{diagram} = \text{diagram}$

$\otimes : (V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V \otimes W})$

WITH $c_x^{V \otimes W} := \text{diagram} = \text{diagram}$

$c : (V, c^V) \otimes (W, c^W) \xrightarrow{c_V^W} (W, c^W) \otimes (V, c^V)$

$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

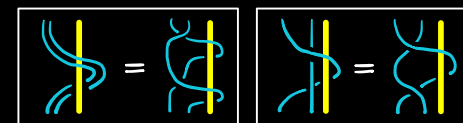
TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

OBJS: $(M \in \mathcal{M}, e^M := \{e_x^M : X \triangleright M \xrightarrow{\sim} X \triangleright M\}_{X \in \mathcal{C}})$

WITH $e_{x \otimes y}^M$ SATISFYING ...

$(\mathcal{M}, \triangleright, e)$



DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

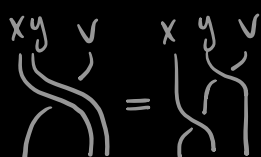
CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

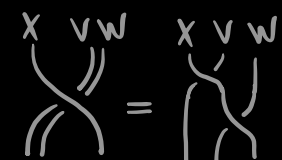
TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING 

$\otimes : (V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V \otimes W})$

WITH $c_x^{V \otimes W} :=$ 

$c : (V, c^V) \otimes (W, c^W) \xrightarrow{c_V^W} (W, c^W) \otimes (V, c^V)$

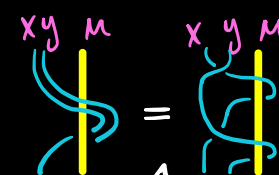
$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

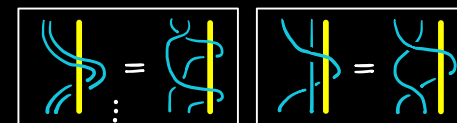
TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

OBJS: $(M \in \mathcal{M}, e^M := \{e_x^M : X \triangleright M \xrightarrow{\sim} X \triangleright M\}_{X \in \mathcal{C}})$

WITH $e_{x \otimes y}^M$ SATISFYING 

$(\mathcal{M}, \triangleright, e)$



DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

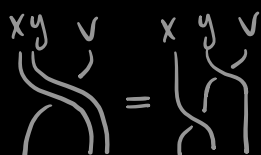
CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

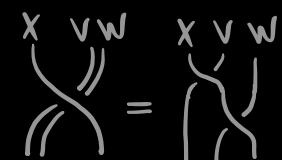
TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING 

$\otimes : (V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V \otimes W})$

WITH $c_x^{V \otimes W} :=$ 

$c : (V, c^V) \otimes (W, c^W) \xrightarrow{c_V^W} (W, c^W) \otimes (V, c^V)$

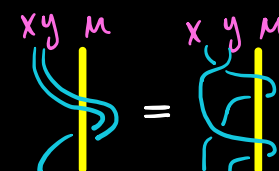
$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

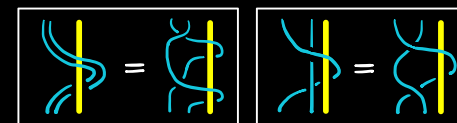
OBJS: $(M \in \mathcal{M}, e^M := \{e_x^M : X \triangleright M \xrightarrow{\sim} X \triangleright M\}_{X \in \mathcal{C}})$

WITH $e_{x \otimes y}^M$ SATISFYING 

$\triangleright : Y \triangleright (M, e^M) := (Y \triangleright M, e^{Y \triangleright M})$

WITH $e_x^{Y \triangleright M} := \dots$

$(\mathcal{M}, \triangleright, e)$



DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

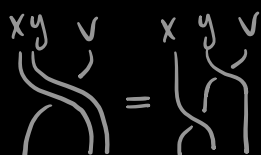
CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

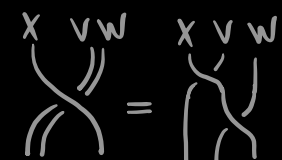
TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

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WITH $c_x^{V \otimes W} :=$ 

$c : (V, c^V) \otimes (W, c^W) \xrightarrow{c_V^W} (W, c^W) \otimes (V, c^V)$

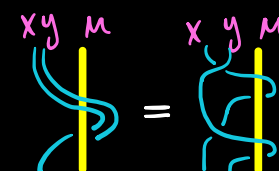
$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

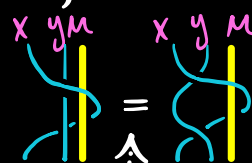
TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

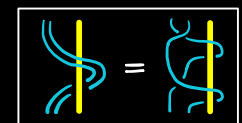
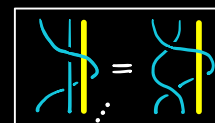
OBJS: $(M \in \mathcal{M}, e^M := \{e_x^M : X \triangleright M \xrightarrow{\sim} X \triangleright M\}_{X \in \mathcal{C}})$

WITH $e_{x \otimes y}^M$ SATISFYING 

$\triangleright : Y \triangleright (M, e^M) := (Y \triangleright M, e^{Y \triangleright M})$

WITH $e_x^{Y \triangleright M} :=$ 

$(\mathcal{M}, \triangleright, e)$

DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

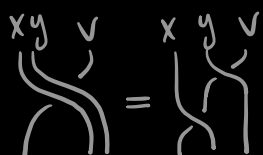
CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

WITH $c_{x \otimes y}^V$ SATISFYING 

$\otimes : (V, c^V) \otimes (W, c^W) := (V \otimes W, c^{V \otimes W})$

WITH $c_x^{V \otimes W} := \img alt="Diagram showing the definition of c_x^{V \otimes W}. It is the composition of two crossings: first x crosses V, then x crosses W." data-bbox="370 601 475 691"/>$

$c : (V, c^V) \otimes (W, c^W) \xrightarrow{c_V^W} (W, c^W) \otimes (V, c^V)$

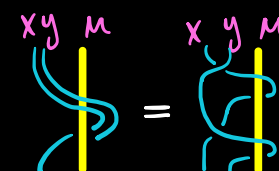
$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

OBJS: $(M \in \mathcal{M}, e^M := \{e_x^M : X \triangleright M \xrightarrow{\sim} X \triangleright M\}_{X \in \mathcal{C}})$

WITH $e_{x \otimes y}^M$ SATISFYING 

$\triangleright : y \triangleright (M, e^M) := (y \triangleright M, e^{y \triangleright M})$

WITH $e_x^{y \triangleright M} := \img alt="Diagram showing the definition of e_x^{y \triangleright M}. It is the composition of two crossings: first x crosses y, then x crosses M." data-bbox="810 590 904 680"/>$

$e : y \triangleright (M, e^M) \xrightarrow{e_y^M} y \triangleright (M, e^M)$

$\equiv$ BRAIDED $\mathcal{C}$ -MODULE CATEGORY $\equiv$

DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

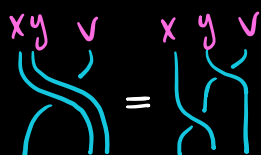
CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

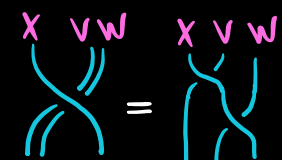
TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

→ DRINFELD CENTER $Z(\mathcal{C})$:

OBJS: $(V \in \mathcal{C}, c^V := \{c_x^V : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

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$c : (V, c^V) \otimes (W, c^W) \xrightarrow{c_V^W} (W, c^W) \otimes (V, c^V)$

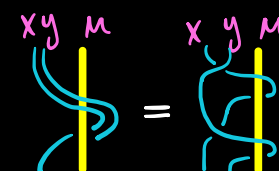
= BRAIDED MONOIDAL CATEGORY =

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

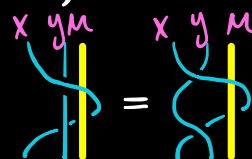
TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

→ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

OBJS: $(M \in \mathcal{M}, e^M := \{e_x^M : X \triangleright M \xrightarrow{\sim} X \triangleright M\}_{X \in \mathcal{C}})$

WITH $e_{x \otimes y}^M$ SATISFYING 

$\triangleright : y \triangleright (M, e^M) := (y \triangleright M, e^{y \triangleright M})$

WITH $e_x^{y \triangleright M} :=$ 

$e : y \triangleright (M, e^M) \xrightarrow{e_y^M} y \triangleright (M, e^M)$

= BRAIDED $\mathcal{C}$ -MODULE CATEGORY =

DRINFELD (1987)
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(1995)

CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $Z(\mathcal{C})$:

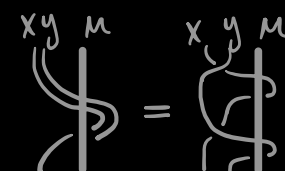
$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

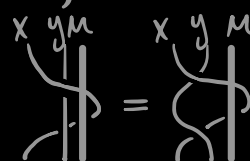
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$\equiv$ BRAIDED $\mathcal{C}$ -MODULE CATEGORY $\equiv$

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CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $\mathcal{Z}(\mathcal{C})$:

$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

HOPF CASE: TAKE F.D. HOPF ALG H .

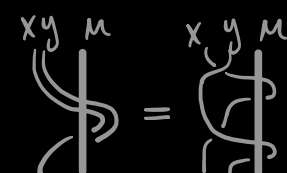
$\rightarrow \boxed{\mathcal{Z}(H\text{-Mod}) \stackrel{\text{BR} \otimes}{\simeq} ???\text{-Mod}}$

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

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WITH $e_x^{y \triangleright M} := \img alt="Diagrammatic equation for e_x^{y \triangleright M}. On the left, a vertical line with a loop labeled x and a line labeled y \triangleright M. On the right, a similar diagram with a different loop configuration." data-bbox="810 589 903 680"/>$

$e : y \triangleright (M, e^M) \xrightarrow{e_y^M} y \triangleright (M, e^M)$

$\equiv$ BRAIDED $\mathcal{C}$ -MODULE CATEGORY $\equiv$

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(1995)

CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $\mathcal{Z}(\mathcal{C})$:

$\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

HOPF CASE: TAKE F.D. HOPF ALG H .

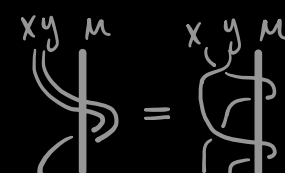
$\rightarrow \boxed{\mathcal{Z}(H\text{-Mod}) \stackrel{\text{BR} \otimes}{\simeq} D(H)\text{-Mod}}$
(QUASI Δ) DRINFELD DOUBLE OF H
($= H^* \otimes H$ AS A VS)

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

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$\triangleright : y \triangleright (M, e^M) := (y \triangleright M, e^{y \triangleright M})$
WITH $e_x^{y \triangleright M} := \img alt="Diagrammatic equation for e_x^{y \triangleright M}. On the left, a vertical line with a crossing labeled x and a cap labeled y \otimes M. On the right, a similar diagram with a crossing labeled x and a cap labeled y \otimes M." data-bbox="810 590 916 680"/>$

$e : y \triangleright (M, e^M) \xrightarrow{e_y^M} y \triangleright (M, e^M)$
 $\equiv$ BRAIDED $\mathcal{C}$ -MODULE CATEGORY $\equiv$

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CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $\mathcal{Z}(\mathcal{C})$:
 $\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

HOPF CASE: TAKE F.D. HOPF ALG H .

$\rightarrow \boxed{\mathcal{Z}(H\text{-Mod}) \stackrel{\text{BR} \otimes}{\simeq} D(H)\text{-Mod}}$
(QUASI Δ) DRINFELD DOUBLE OF H
($= H^* \otimes H$ AS A VS)

SHIMIZU: H, H' F.D. HOPF ALGS.

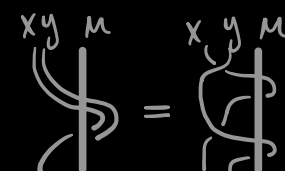
$H\text{-Mod} \simeq H'\text{-Mod} \Rightarrow \rho_n^{D(H)} \simeq \rho_n^{D(H')}$
AS MONOIDAL CATS AS REPS OF Br_{n-1}^A

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:

OBJS: $(M \in \mathcal{M}, e^M := \{e_x^M : X \triangleright M \xrightarrow{\sim} X \triangleright M\}_{x \in \mathcal{C}})$

WITH $e_{x \otimes y}^M$ SATISFYING 

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WITH $e_x^{y \triangleright M} := \img alt="Diagrammatic equation for e_x^{y \triangleright M}. On the left, a vertical line with a loop labeled x and a line labeled y \otimes M. On the right, a similar diagram with a loop labeled x and a line labeled y \otimes M." data-bbox="810 590 916 680"/>$

$e : y \triangleright (M, e^M) \xrightarrow{e_y^M} y \triangleright (M, e^M)$
 $\equiv$ BRAIDED $\mathcal{C}$ -MODULE CATEGORY $\equiv$

DRINFELD (1987)
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CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $\mathcal{Z}(\mathcal{C})$:
 $\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

HOPF CASE: TAKE F.D. HOPF ALG H .

$\rightarrow \boxed{\mathcal{Z}(H\text{-Mod}) \stackrel{\text{BR} \otimes}{\simeq} D(H)\text{-Mod}}$
 $\uparrow$
 (QUASI Δ) DRINFELD DOUBLE OF H
 ($= H^* \otimes H$ AS A VS)

SHIMIZU: H, H' F.D. HOPF ALGS.

$H\text{-Mod} \simeq H'\text{-Mod} \Rightarrow \rho_n^{D(H)} \simeq \rho_n^{D(H')}$
 AS MONOIDAL CATS AS REPS OF Br_{n-1}^A

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

$\rightarrow$ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:
 $\equiv$ BRAIDED $\mathcal{C}$ -MODULE CATEGORY $\equiv$

DRINFELD (1987)
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(1995)

CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

→ DRINFELD CENTER $\mathcal{Z}(\mathcal{C})$:
= BRAIDED MONOIDAL CATEGORY =

HOPF CASE: TAKE F.D. HOPF ALG H .

→ $\mathcal{Z}(H\text{-Mod}) \stackrel{\text{BR} \otimes}{\simeq} D(H)\text{-Mod}$
(QUASI Δ) DRINFELD DOUBLE OF H
(= $H^* \otimes H$ AS A VS)

SHIMIZU: H, H' F.D. HOPF ALGS.

$H\text{-Mod} \simeq H'\text{-Mod} \Rightarrow \rho_n^{D(H)} \simeq \rho_n^{D(H')}$
AS MONOIDAL CATS AS REPS OF Br_{n-1}^A

FIX A BRAIDED MONOIDAL CAT $\mathcal{C}$

TAKE A $\mathcal{C}$ -MODULE CAT $(\mathcal{M}, \triangleright)$

→ REFLECTIVE CENTER $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$:
= BRAIDED $\mathcal{C}$ -MODULE CATEGORY =

HOPF CASE: FIX A F.D. QUASI Δ HOPF ALG H
& TAKE A F.D. H -COMOD ALG A

→ $\mathcal{E}_{H\text{-Mod}}(A\text{-Mod}) \stackrel{\text{BR MOD}}{\simeq} ???\text{-Mod}$

DRINFELD (1987)
JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

CATEGORICAL CENTERS

LAUGWITZ-W
-YAKIMOV (2025)

TAKE A MONOIDAL CAT $(\mathcal{C}, \otimes, \mathbb{1})$

$\rightarrow$ DRINFELD CENTER $\mathcal{Z}(\mathcal{C})$:
 $\equiv$ BRAIDED MONOIDAL CATEGORY $\equiv$

HOPF CASE: TAKE F.D. HOPF ALG H .

$\rightarrow \boxed{\mathcal{Z}(H\text{-Mod}) \stackrel{\text{BR} \otimes}{\simeq} D(H)\text{-Mod}}$
 $\uparrow$
(QUASI Δ) DRINFELD DOUBLE OF H
($= H^* \otimes H$ AS A VS)

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& TAKE A F.D. H -COMOD ALG A

$\rightarrow \boxed{\mathcal{E}_{H\text{-Mod}}(A\text{-Mod}) \stackrel{\text{BR MOD}}{\simeq} R_H(A)\text{-Mod}}$
 $\uparrow$
(QUASI Δ) REFLECTIVE ALG OF A W.R.T. H
($= H^* \otimes A$ AS A VS)

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JOYAL-STREET (1991)
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& TAKE A F.D. H -COMOD ALG A

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($= H^* \otimes A$ AS A VS)

MÜLLER-W: A, A' COIDEAL SUBALGS OF H .

$A\text{-Mod} \simeq A'\text{-Mod} \Rightarrow \rho_n^{R_H(A)} \simeq \rho_n^{R_H(A')}$
AS $(H\text{-Mod})$ -MODULE CATS AS REPS OF Br_n^B

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JOYAL-STREET (1991)
MAJID (1991)

KASSEL
(1995)

CATEGORICAL CENTERS

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→ $\mathcal{Z}(H\text{-Mod}) \stackrel{\text{BR} \otimes}{\simeq} D(H)\text{-Mod}$
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(= $H^* \otimes H$ AS A VS)

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$\equiv \text{TYPE A} \equiv$

SUMMARY ... SO FAR

$\equiv \text{TYPE B}^{\mathbb{C}} \equiv$

HOPF CASE:

QUASIA HOPF ALGS

... UP TO BR. MORITA EQUIV

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

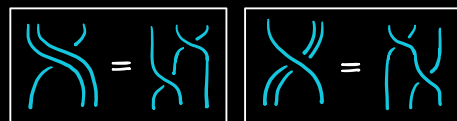
QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

OBJECTS

FOR GENERAL
REP'NS OF Br^A

BRAIDED
MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$



BRAID GROUP OF TYPE A

$$\rho_n^X: Br_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \longmapsto \begin{array}{c} \text{diagram of a crossing between strands } i \text{ and } i+1 \end{array}$$

HOPF CASE:

QUASIA COMODULE ALGS

... UP TO BR. MORITA EQUIV

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

FOR GENERAL
REP'NS OF Br^B

OBJECTS

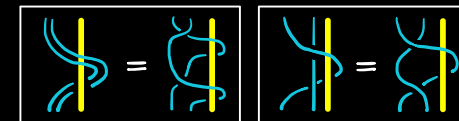
OBJECTS

BRAID GROUP OF TYPE B

$$\rho_n^{X,M}: Br_n^B \longrightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$$

$$\begin{array}{l} \sigma_i \longmapsto \text{diagram of a crossing between strands } i \text{ and } i+1 \\ t \longmapsto \text{diagram of a twist on strand } i \end{array}$$

BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



$\equiv \text{TYPE A} \equiv$

SUMMARY ... SO FAR

$\equiv \text{TYPE B}^{\mathbb{C}} \equiv$

HOPF CASE:
QUASIA HOPF ALGS
... UP TO BR. MORITA EQUIV

CATEG. CONSTR'N:
DRINFELD
CENTER

CATEG. CONSTR'N:
REFLECTIVE
CENTER

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$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

OBJECTS

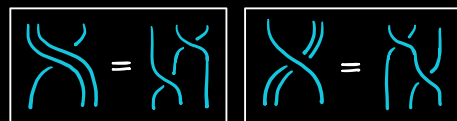
FOR GENERAL
REP'NS OF Br^A

FOR GENERAL
REP'NS OF Br^B

OBJECTS

OBJECTS

BRAIDED
MONOIDAL CATEGORY
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$$\rho_n^X: Br_{n-1}^A \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \longmapsto \text{diagram of two strands crossing at positions } i \text{ and } i+1$$

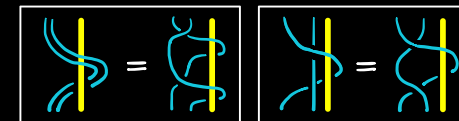
BRAID GROUP OF TYPE B

$$\rho_n^{X,M}: Br_n^B \longrightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$$

$$\sigma_i \longmapsto \text{diagram of two strands crossing with a vertical line},$$

$$t \longmapsto \text{diagram of a strand with a loop}$$

BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



$\equiv \text{Type A} \equiv$

SUMMARY ... SO FAR

$\equiv \text{Type B}^{\mathbb{C}} \equiv$

DRINFELD
DOUBLE

REFLECTIVE
ALGS.

CATEG. CONSTR'N:
DRINFELD
CENTER

CATEG. CONSTR'N:
REFLECTIVE
CENTER

HOPF CASE:
QUASIA HOPF ALGS
... UP TO BR. MORITA EQUIV

HOPF CASE:
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$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

OBJECTS

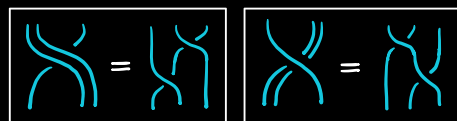
FOR GENERAL
REP'NS OF Br^A

FOR GENERAL
REP'NS OF Br^B

OBJECTS

OBJECTS

BRAIDED
MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$



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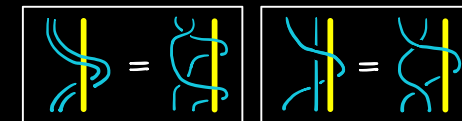
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BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



$\equiv \text{TYPE A} \equiv$

ON NONDEGENERACY

$\equiv \text{TYPE B}^c \equiv$

DRINFELD
DOUBLE

HOPF CASE:

QUASIA HOPF ALGS

... UP TO BR. MORITA EQUIV

CATEG. CONSTRIN:

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CENTER

REFLECTIVE
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CATEG. CONSTRIN:

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CENTER

HOPF CASE:

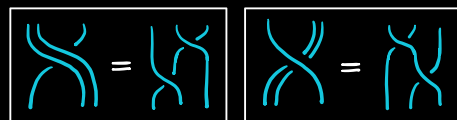
QUASIA COMODULE ALGS

... UP TO BR. MORITA EQUIV

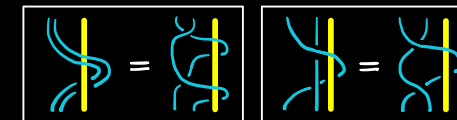
TOWARDS "MODULARITY"

... USED IN APPLICATIONS [3D-TQFTS, TQC, ...]

BRAIDED
MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, 1, c)$



BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



SHIMIZU (2019)

BUILDING ON SEVERAL WORKS
IN THE SEMISIMPLE CASE

ON NONDEGENERACY

THEOREM: GIVEN A BRAIDED MONOIDAL CAT.

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ THAT IS "FINITE TENSOR"

[LIKE $H\text{-FdMod}$ FOR H F.D. QUASITRIANGULAR HOPF ALG.]

TFAE:

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TFAE:

(1)

(2)

$$(3) \mathcal{I}_2(\mathcal{C}) := \left\{ x \in \mathcal{C} : \sum_{x \otimes y} x \otimes y = \prod_{x \otimes y} x \otimes y \quad \forall y \in \mathcal{C} \right\} \simeq \text{Vec}$$

$\equiv \mathcal{C}$ HAS TRIVIAL SYM. CENTER

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(1)

$$(2) \mathcal{Z}(\mathcal{C}) \simeq (\mathcal{C}, c) \boxtimes (\mathcal{C}, c^{-1}) \\ \equiv \mathcal{C} \text{ IS FACTORIZABLE}$$

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FACTORIZABLE?

TFAE:

$$(1) \text{ COEND } \int_{X \in \mathcal{C}} X^* \otimes X \in \text{HopfAlg}(\mathcal{C})$$

HAS A NONDEGENERATE HOPF PAIRING

$$(2) \mathcal{Z}(\mathcal{C}) \simeq (\mathcal{C}, c) \boxtimes (\mathcal{C}, c^{-1})$$

$\equiv \mathcal{C}$ IS FACTORIZABLE

$$(3) \mathcal{I}_2(\mathcal{C}) := \left\{ X \in \mathcal{C} : \sum_{\substack{x \otimes y \\ x \otimes y}}^{\substack{x \otimes y \\ x \otimes y}} = \prod_{\substack{x \otimes y \\ x \otimes y}} \forall y \in \mathcal{C} \right\} \simeq \text{Vec}$$

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Ex. $\mathcal{D}(H)\text{-FdMod}$, $u_q(\mathfrak{g})\text{-FdMod}$

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BUILDING ON SEVERAL WORKS
IN THE SEMISIMPLE CASE

ON NONDEGENERACY

W-YADAV (2025)

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FIX BRAIDED FINITE TENSOR CAT. $\mathcal{C}$

THEOREM: GIVEN A BRAIDED $\mathcal{C}$ -MODULE CAT.

$(\mathcal{M}, \triangleright, e)$ THAT IS "EXACT".

GET:

(1)

$\Updownarrow$

(2)

$\Downarrow$

(3)

SHIMIZU (2019)

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(3) $\mathcal{Z}_2(\mathcal{C}) := \left\{ X \in \mathcal{C} : \sum_{x,y} \begin{array}{c} x \ y \\ \diagdown \ \diagup \\ \diagup \ \diagdown \\ x \ y \end{array} = \begin{array}{c} x \ y \\ \diagdown \ \diagup \\ \diagup \ \diagdown \\ x \ y \end{array} \forall y \in \mathcal{C} \right\} \simeq \text{Vec}$
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GET:

(1) END $\int_{M \in \mathcal{M}} \underline{\text{Hom}}(M, M) \in \text{Alg}(\mathcal{C})$



HAS A NONDEGENERATE PAIRING

(2) $\mathcal{E}_{\mathcal{C}}(\mathcal{M}) \simeq \mathcal{M} \boxtimes \text{Func}(\mathcal{M}, \mathcal{M})$



$\equiv \mathcal{M}$ IS FACTORIZABLE

(3) $\mathcal{Z}_2(\mathcal{M}) := \left\{ X \in \mathcal{C} : \begin{array}{c} x \ M \\ \diagdown \ \diagup \\ \diagup \ \diagdown \\ x \ M \end{array} = \begin{array}{c} x \ M \\ \diagdown \ \diagup \\ \diagup \ \diagdown \\ x \ M \end{array} \forall M \in \mathcal{M} \right\} \simeq \text{Vec}$
 $\equiv \mathcal{M}$ HAS TRIVIAL SYM. CENTER

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 $\equiv \mathcal{C}$ HAS TRIVIAL SYM. CENTER

Ex. $\mathcal{D}(H)\text{-FdMod}$, $u_q(\mathfrak{g})\text{-FdMod}$

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GET:

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$\Updownarrow$

HAS A NONDEGENERATE PAIRING

(2) $\mathcal{E}_{\mathcal{C}}(\mathcal{M}) \simeq \mathcal{M} \boxtimes \text{Func}(\mathcal{M}, \mathcal{M})$

$\Downarrow$

$\equiv \mathcal{M}$ IS FACTORIZABLE

(3) $\mathcal{Z}_2(\mathcal{M}) := \left\{ X \in \mathcal{C} : \begin{array}{c} x \ M \\ \diagdown \ \diagup \\ \diagup \ \diagdown \\ x \ M \end{array} = \begin{array}{c} x \ M \\ \diagup \ \diagdown \\ \diagdown \ \diagup \\ x \ M \end{array} \forall M \in \mathcal{M} \right\} \simeq \text{Vec}$
 $\equiv \mathcal{M}$ HAS TRIVIAL SYM. CENTER

Ex. $R_{\#}(A)\text{-FdMod}$, QUANTUM GROUP EXS?

SHIMIZU (2019)

BUILDING ON SEVERAL WORKS
IN THE SEMISIMPLE CASE

ON NONDEGENERACY

W-YADAV (2025)

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[LIKE $H\text{-FdMod}$ FOR H F.D. QUASIA HOPF ALG.]
FACTORIZABLE?

TFAE:

(1) $\text{coEND} \int^{X \in \mathcal{C}} X^* \otimes X \in \text{HopfAlg}(\mathcal{C})$
HAS A NONDEGENERATE HOPF PAIRING

(2) $\mathcal{Z}(\mathcal{C}) \simeq (\mathcal{C}, c) \boxtimes (\mathcal{C}, c^{-1})$
 $\equiv \mathcal{C}$ IS FACTORIZABLE

(3) $\mathcal{Z}_2(\mathcal{C}) := \left\{ X \in \mathcal{C} : \begin{array}{c} x \ y \\ \text{---} \\ x \ y \end{array} = \begin{array}{c} x \ y \\ \text{---} \\ x \ y \end{array} \forall y \in \mathcal{C} \right\} \simeq \text{Vec}$
 $\equiv \mathcal{C}$ HAS TRIVIAL SYM. CENTER

Ex. $\mathcal{D}(H)\text{-FdMod}$, $u_q(\mathfrak{g})\text{-FdMod}$

FIX BRAIDED FINITE TENSOR CAT. $\mathcal{C}$

THEOREM: GIVEN A BRAIDED $\mathcal{C}$ -MODULE CAT.

$(\mathcal{M}, \triangleright, e)$ THAT IS "EXACT".

GET:

(1) $\text{END} \int_{M \in \mathcal{M}} \underline{\text{Hom}}(M, M) \in \text{Alg}(\mathcal{C})$

$\Updownarrow$

HAS A NONDEGENERATE PAIRING

(2) $\mathcal{E}_{\mathcal{C}}(\mathcal{M}) \simeq \mathcal{M} \boxtimes \text{Func}(\mathcal{M}, \mathcal{M})$

$\Downarrow$

$\equiv \mathcal{M}$ IS FACTORIZABLE

(3) $\mathcal{Z}_2(\mathcal{M}) := \left\{ X \in \mathcal{C} : \begin{array}{c} x \ M \\ \text{---} \\ x \ M \end{array} = \begin{array}{c} x \ M \\ \text{---} \\ x \ M \end{array} \forall M \in \mathcal{M} \right\} \simeq \text{Vec}$
 $\equiv \mathcal{M}$ HAS TRIVIAL SYM. CENTER

Ex. $R_{\#}(A)\text{-FdMod}$, QUANTUM GROUP EXS?

$\equiv \text{Type A} \equiv$

SUMMARY

$\equiv \text{Type B} \stackrel{C}{=} \equiv$

DRINFELD
DOUBLE

CATEG. CONSTR'N:
DRINFELD CENTER

HOPF CASE:

QUASIA HOPF ALGS

... UP TO BR. MORITA EQUIV

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

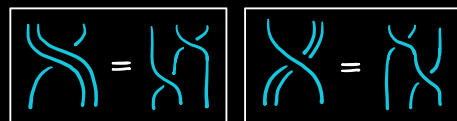
QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

OBJECTS

OBJECTS

FOR GENERAL
REP'NS OF Br^A

BRAIDED
MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, 1, c)$



BRAID GROUP OF TYPE A

$$\rho_n^X: Br_{n-1}^A \rightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \mapsto \begin{array}{c} \text{diagram of } \sigma_i \text{ in } Br_n^A \end{array}$$

CATEG. CONSTR'N:
REFLECTIVE CENTER

REFLECTIVE
ALGS.

HOPF CASE:

QUASIA COMODULE ALGS

... UP TO BR. MORITA EQUIV

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

QUANTUM REFLECTION EQ (qRE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

FOR GENERAL
REP'NS OF Br^B

OBJECTS

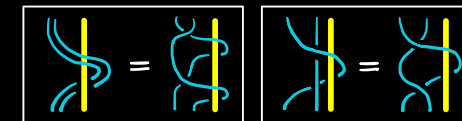
OBJECTS

BRAID GROUP OF TYPE B

$$\rho_n^{X,M}: Br_n^B \rightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n} \triangleright M)$$

$$\begin{array}{l} \sigma_i \mapsto \text{diagram of } \sigma_i \text{ in } Br_n^B \\ t \mapsto \text{diagram of } t \text{ in } Br_n^B \end{array}$$

BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



$\equiv \text{Type A} \equiv$

SUMMARY

$\equiv \text{Type B} \stackrel{C}{=} \equiv$

DRINFELD DOUBLE
 CATEG. CONSTR'N:
 DRINFELD CENTER
 HOPF CASE: $\swarrow$
 QUASIA HOPF ALGS
 ... UP TO BR. MORITA EQUIV
 NONDEGENERACY

REFLECTIVE ALGS.
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 QUASIA COMODULE ALGS
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$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

QUANTUM YANG-BAXTER EQ. (qYBE)
 $\equiv \text{WANT SOLUTIONS} \equiv$

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QUANTUM REFLECTION EQ (qRE)
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OBJECTS

OBJECTS

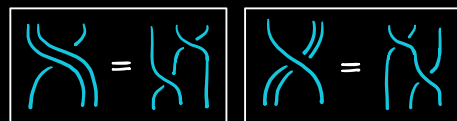
FOR GENERAL
 REPR'S OF Br^A

FOR GENERAL
 REPR'S OF Br^B

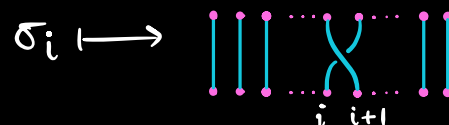
OBJECTS

OBJECTS

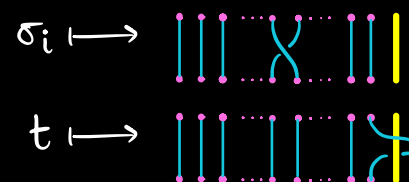
BRAIDED
 MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$



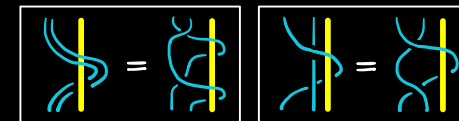
BRAID GROUP OF TYPE A
 $\rho_n^X: Br_{n-1}^A \rightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$



BRAID GROUP OF TYPE B
 $\rho_n^{X,M}: Br_n^B \rightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$



BRAIDED
 $\mathcal{C}$ -MODULE CATEGORY
 $(\mathcal{M}, \triangleright, e)$



$\equiv \text{TYPE A} \equiv$

QUANTUM
YANG-BAXTER
EQ. (qYBE)

$$R_{12} R_{13} R_{23} \\ = R_{23} R_{13} R_{12}$$

BRAIDED
MONOIDAL CAT
 $(\mathcal{C}, \otimes, \mathbb{1}, c)$

$$\rho_n^X: Br_{n-1}^A \curvearrowright \\ \text{Aut}_e(X^{\otimes n})$$

$$\sigma_i \mapsto$$

$\equiv \text{SPECULATIONS} \equiv$

$\equiv \text{TYPE B} \stackrel{c}{=} \equiv$

QUANTUM
REFLECTION
EQ (qRE)

$$R_{12} K_1 R_{21} K_2 \\ = K_2 R_{12} K_1 R_{21}$$

BRAIDED
 $\mathcal{C}$ -MODULE CAT
 $(\mathcal{M}, \triangleright, e)$

$$\rho_n^{X,M}: Br_n^B \curvearrowright \\ \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$$

$$\sigma_i \mapsto$$

$$t \mapsto$$

≡ TYPE A ≡

QUANTUM
YANG-BAXTER
EQ. (q YBE)

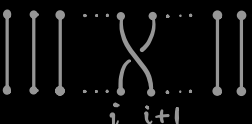
$$R_{12} R_{13} R_{23} \\ = R_{23} R_{13} R_{12}$$

BRAIDED
MONOIDAL CAT
($\mathcal{C}, \otimes, \mathbb{1}, c$)

$$\text{Diagram 1} = \text{Diagram 2}$$

$$\text{Diagram 3} = \text{Diagram 4}$$

$$\rho_n^X: Br_{n-1}^A \hookrightarrow \\ Aut_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \mapsto$$


≡ SPECULATIONS ≡

THERE ARE LOADS OF COXETER TYPES ...

≡ TYPE B^C ≡

QUANTUM
REFLECTION
EQ (q RE)

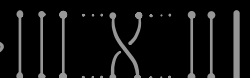
$$R_{12} K_1 R_{21} K_2 \\ = K_2 R_{12} K_1 R_{21}$$

BRAIDED
 $\mathcal{C}$ -MODULE CAT
($\mathcal{M}, \triangleright, e$)

$$\text{Diagram 5} = \text{Diagram 6}$$

$$\text{Diagram 7} = \text{Diagram 8}$$

$$\rho_n^{X,M}: Br_n^B \hookrightarrow \\ Aut_{\mathcal{M}}(X^{\otimes n} \triangleright M)$$

$$\sigma_i \mapsto$$


$$t \mapsto$$


$\equiv \text{TYPE A} \equiv$

QUANTUM
YANG-BAXTER
EQ. (qYBE)

$$R_{12} R_{13} R_{23} \\ = R_{23} R_{13} R_{12}$$

BRAIDED
MONOIDAL CAT
($\mathcal{C}, \otimes, \mathbb{1}, c$)

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

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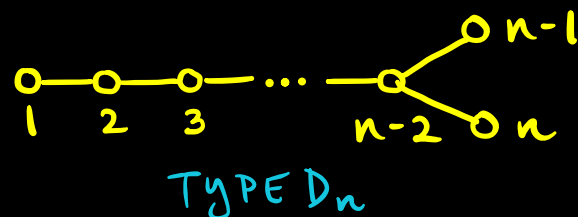
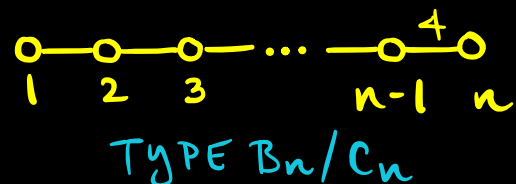
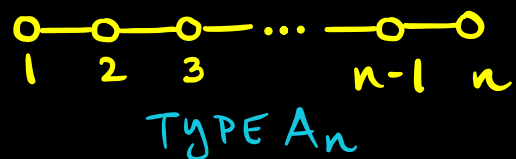
$$\rho_n^X: Br_{n-1}^A \hookrightarrow \\ \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \mapsto \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$\equiv \text{SPECULATIONS} \equiv$

THERE ARE LOADS OF COXETER TYPES ...

COXETER GRAPHS



$E_6 \ E_7 \ E_8 \ F_4 \ G_2$

$H_3 \ H_4 \ I_2(m)$

AFFINE TYPES...

$\equiv \text{TYPE B}^c \equiv$

QUANTUM
REFLECTION
EQ (qRE)

$$R_{12} K_1 R_{21} K_2 \\ = K_2 R_{12} K_1 R_{21}$$

BRAIDED
 $\mathcal{C}$ -MODULE CAT
($\mathcal{M}, \triangleright, e$)

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$\begin{array}{c} \text{ } \\ \text{ } \end{array} = \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

$$\rho_n^{X,M}: Br_n^B \hookrightarrow \\ \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$$

$$\sigma_i \mapsto \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

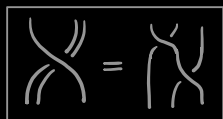
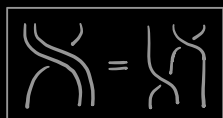
$$t \mapsto \begin{array}{c} \text{ } \\ \text{ } \end{array}$$

≡ TYPE A ≡

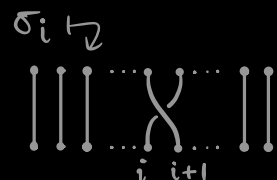
QUANTUM
YANG-BAXTER
EQ. (qYBE)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

BRAIDED
MONOIDAL CAT
($\mathcal{C}, \otimes, \mathbb{1}, c$)



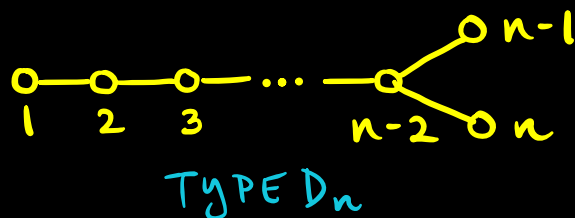
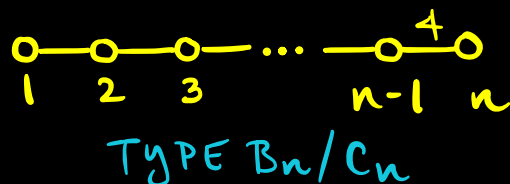
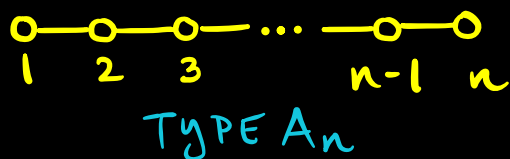
$$\rho_n^X: Br_{n-1}^A \hookrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$



≡ SPECULATIONS ≡

THERE ARE LOADS OF COXETER TYPES...

COXETER GRAPHS



$E_6 \ E_7 \ E_8 \ F_4 \ G_2$

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AFFINE TYPES...

COXETER GROUPS
IN TYPE X_n

$$Br_n^X: \text{GENS: } \theta_1, \dots, \theta_n$$

RELATIONS:

$$\underbrace{\theta_i \theta_j \theta_i \dots}_{a_{ij} \text{ FACTORS}} = \underbrace{\theta_j \theta_i \theta_j \dots}_{a_{ij} \text{ FACTORS}}$$

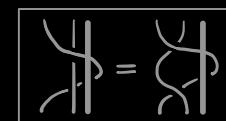
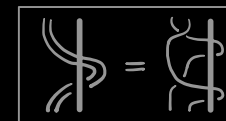
FOR (a_{ij}) COXETER
MATRIX OF TYPE X_n

≡ TYPE B^C ≡

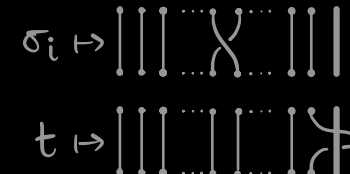
QUANTUM
REFLECTION
EQ (qRE)

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BRAIDED
 $\mathcal{C}$ -MODULE CAT
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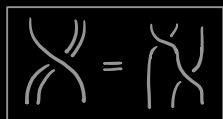
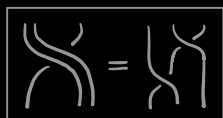


≡ TYPE A ≡

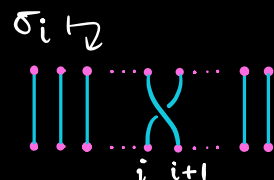
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BRAIDED
MONOIDAL CAT
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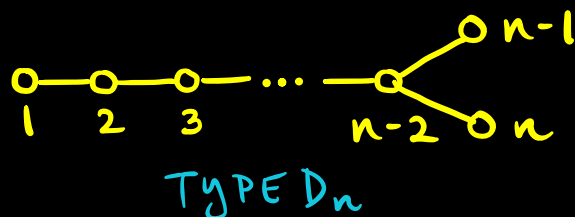
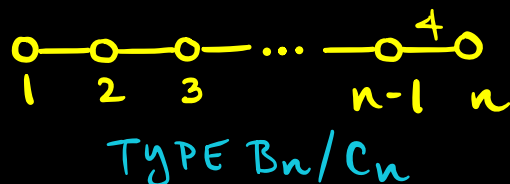
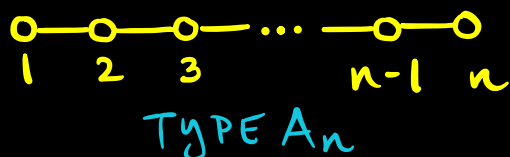
$$\rho_n^X: Br_{n-1}^A \hookrightarrow Aut_{\mathcal{C}}(X^{\otimes n})$$



≡ SPECULATIONS ≡

THERE ARE LOADS OF COXETER TYPES...

COXETER GRAPHS



E_6 E_7 E_8 F_4 G_2

H_3 H_4 $I_2(m)$

AFFINE TYPES...

COXETER GROUPS
IN TYPE X_n

$$Br_n^X: \text{GENS: } \theta_1, \dots, \theta_n$$

RELATIONS:

$$\theta_i \theta_j \theta_i \dots = \theta_j \theta_i \theta_j \dots$$

a_{ij} FACTORS

a_{ij} FACTORS

FOR (a_{ij}) COXETER
MATRIX OF TYPE X_n

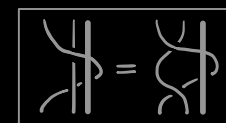
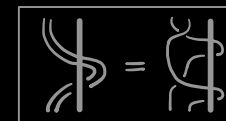
CAN WE REPRESENT
THESE GROUPS W/
STRING MODELS??

≡ TYPE B^c ≡

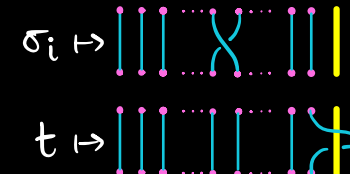
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BRAIDED
 $\mathcal{C}$ -MODULE CAT
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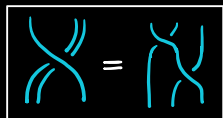
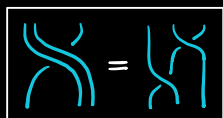


≡ TYPE A ≡

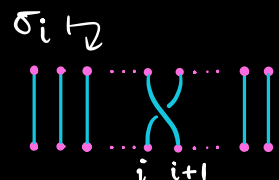
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BRAIDED
MONOIDAL CAT
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≡ SPECULATIONS ≡

THERE ARE LOADS OF COXETER TYPES ...

Br_n^X : GENS: $\theta_1, \dots, \theta_n$

RELATIONS:

$$\underbrace{\theta_i \theta_j \theta_i \dots}_{a_{ij} \text{ FACTORS}} = \underbrace{\theta_j \theta_i \theta_j \dots}_{a_{ij} \text{ FACTORS}}$$

FOR (a_{ij}) COXETER
MATRIX OF TYPE X_n

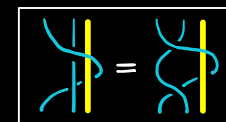
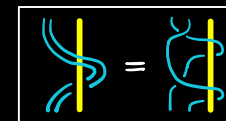
CAN WE REPRESENT
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≡ TYPE B^C ≡

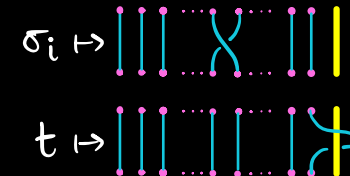
QUANTUM
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 $\mathcal{C}$ -MODULE CAT
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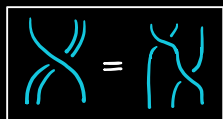
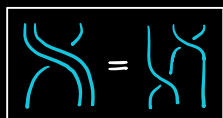


≡ TYPE A ≡

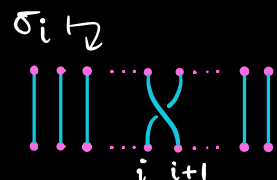
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BRAIDED
MONOIDAL CAT
($\mathcal{C}, \otimes, \mathbb{1}, c$)



$\rho_n^X: Br_{n-1}^A \hookrightarrow \text{Aut}_e(X^{\otimes n})$



≡ SPECULATIONS ≡

THERE ARE LOADS OF COXETER TYPES ...

IS THERE A
BRAIDED
CATEGORICAL
STRUCTURE
"OF TYPE X"
WHOSE
OBJECTS YIELD
REPS OF Br_n^X ??

Br_n^X : GENS: $\theta_1, \dots, \theta_n$

RELATIONS:

$$\underbrace{\theta_i \theta_j \theta_i \dots}_{a_{ij} \text{ FACTORS}} = \underbrace{\theta_j \theta_i \theta_j \dots}_{a_{ij} \text{ FACTORS}}$$

FOR (a_{ij}) COXETER
MATRIX OF TYPE X_n

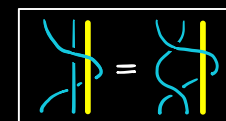
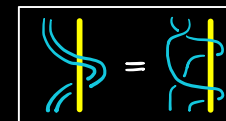
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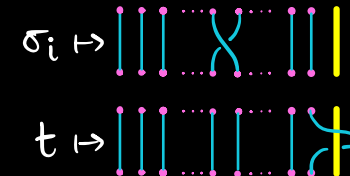
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$\equiv \text{TYPE A} \equiv$

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BRAIDED
MONOIDAL CAT
($\mathcal{C}, \otimes, \mathbb{1}, c$)

$$\text{Crossing} = \text{Braid}$$

$$\text{Crossing} = \text{Braid}$$

$$\rho_n^X: Br_{n-1}^A \curvearrowright \text{Aut}_e(X^{\otimes n})$$

$$\sigma_i \mapsto \text{Diagram with } i \text{ and } i+1 \text{ strands crossing}$$

$\equiv \text{SPECULATIONS} \equiv$

THERE ARE LOADS OF COXETER TYPES ...

IS THERE A
BRAIDED
CATEGORICAL
STRUCTURE
"OF TYPE X"
WHOSE
OBJECTS YIELD
REPIQS OF Br_n^X ??

BRAID GROUP OF
COXETER TYPE X

CAN WE REPRESENT
THESE GROUPS W/
STRING MODELS??

$\equiv \text{TYPE B}^c \equiv$

QUANTUM
REFLECTION
EQ (qRE)

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

BRAIDED
 $\mathcal{C}$ -MODULE CAT
($\mathcal{M}, \triangleright, e$)

$$\text{Crossing} = \text{Braid}$$

$$\text{Crossing} = \text{Braid}$$

$$\rho_n^{X,M}: Br_n^B \curvearrowright \text{Aut}_m(X^{\otimes n} \triangleright M)$$

$$\sigma_i \mapsto \text{Diagram with } i \text{ and } i+1 \text{ strands crossing}$$

$$t \mapsto \text{Diagram with } i \text{ and } i+1 \text{ strands crossing}$$

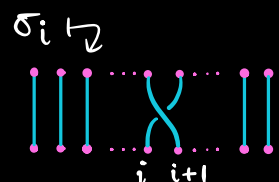
$\equiv \text{TYPE A} \equiv$

QUANTUM
YANG-BAXTER
EQ. (qYBE)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

BRAIDED
MONOIDAL CAT
($\mathcal{C}, \otimes, \mathbb{1}, c$)

$$\rho_n^X: Br_{n-1}^A \curvearrowright \text{Aut}_e(X^{\otimes n})$$



$\equiv \text{SPECULATIONS} \equiv$

CAN WE USE OBJECTS IN
BRAIDED CAT. STRUCTURES
"OF TYPE X" TO MODEL
PARTICLE SCATTERING
IN PHYSICAL SYSTEMS
W/ OBSTRUCTIONS??

IS THERE A
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CATEGORICAL
STRUCTURE
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WHOSE
OBJECTS YIELD
REPIES OF Br_n^X ??

BRAID GROUP OF
COXETER TYPE X

CAN WE REPRESENT
THESE GROUPS W/
STRING MODELS??

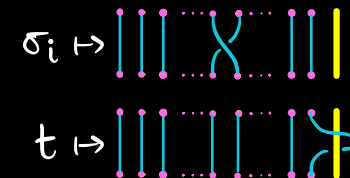
$\equiv \text{TYPE B}^c \equiv$

QUANTUM
REFLECTION
EQ (qRE)

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

BRAIDED
 $\mathcal{C}$ -MODULE CAT
($\mathcal{M}, \triangleright, e$)

$$\rho_n^{X,M}: Br_n^B \curvearrowright \text{Aut}_m(X^{\otimes n} \triangleright M)$$



$\equiv \text{TYPE A} \equiv$

QUANTUM
YANG-BAXTER
EQ. (qYBE)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

BRAIDED
MONOIDAL CAT
($\mathcal{C}, \otimes, \mathbb{1}, c$)

$$\text{Diagram 1} = \text{Diagram 2}$$

$$\text{Diagram 3} = \text{Diagram 4}$$

$$\rho_n^X: Br_{n-1}^A \curvearrowright \text{Aut}_e(X^{\otimes n})$$

$$\sigma_i \mapsto \text{Diagram with crossing at } i, i+1$$

AND IS THERE 1 EQUATION TO RULE THEM ALL??



IS THERE A
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CATEGORICAL
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REFLECTION
EQ (qRE)

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

BRAIDED
 $\mathcal{E}$ -MODULE CAT
($\mathcal{M}, \triangleright, e$)

$$\text{Diagram 5} = \text{Diagram 6}$$

$$\text{Diagram 7} = \text{Diagram 8}$$

$$\rho_n^{X,M}: Br_n^B \curvearrowright \text{Aut}_m(X^{\otimes n} \triangleright M)$$

$$\sigma_i \mapsto \text{Diagram with crossing and vertical lines}$$

$$t \mapsto \text{Diagram with crossing and vertical lines}$$

≡ TYPE A ≡

QUANTUM
YANG-BAXTER
EQ. (qYBE)

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

BRAIDED
MONOIDAL CAT
($\mathcal{C}, \otimes, \mathbb{1}, c$)

$$\text{Diagram 1} = \text{Diagram 2}$$

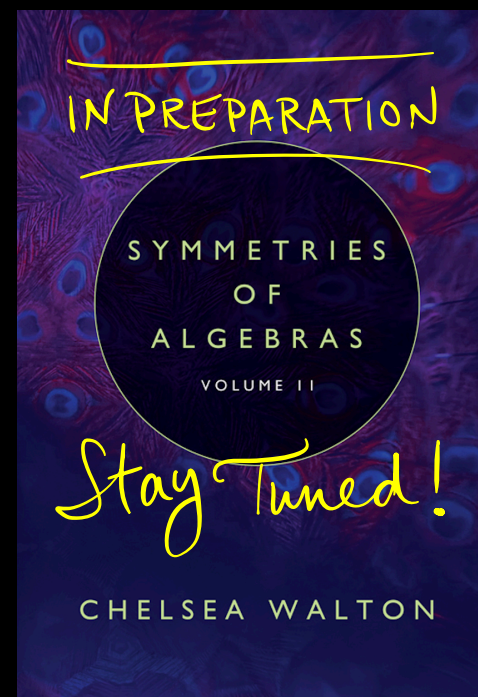
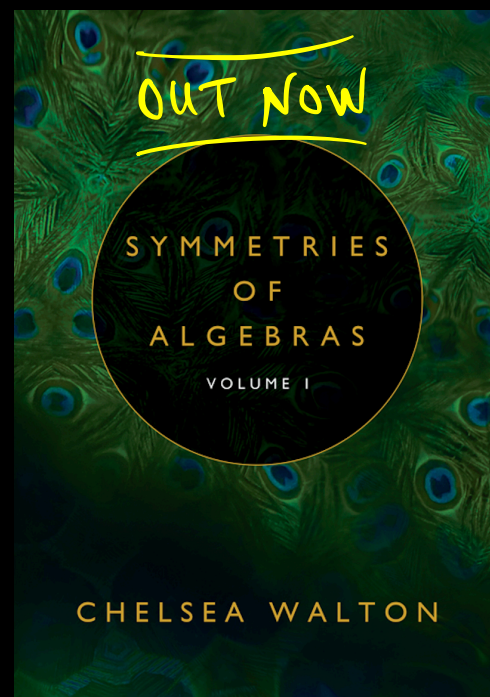
$$\text{Diagram 3} = \text{Diagram 4}$$

$$\rho_n^X: Br_{n-1}^A \hookrightarrow \text{Aut}_e(X^{\otimes n})$$

$$\sigma_i \mapsto \text{Diagram 5}$$

Thanks for listening!!

NOTLAW @ RICE.EDU



≡ TYPE B^{=C} ≡

QUANTUM
REFLECTION
EQ (qRE)

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21}$$

BRAIDED
 $\mathcal{C}$ -MODULE CAT
($\mathcal{M}, \triangleright, e$)

$$\text{Diagram 6} = \text{Diagram 7}$$

$$\text{Diagram 8} = \text{Diagram 9}$$

$$\rho_n^{X,M}: Br_n^B \hookrightarrow \text{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$$

$$\sigma_i \mapsto \text{Diagram 10}$$

$$t \mapsto \text{Diagram 11}$$