Crystallization of function algebras

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Classical $\rightsquigarrow$ quantized enveloping algebras

Ex. $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$,

$U(\mathfrak{g})$ is generated by elements E_i, F_i, H_i,

$$E_i = \begin{pmatrix} 0 & & & \\ & \ddots & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad F_i = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & 1 & \ddots & \\ & & & 0 \end{pmatrix}, \quad H_i = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

with the Chevalley-Serre relations:

$$[H_j, E_i] = \alpha_i(H_j) E_i$$

$$[H_j, F_i] = -\alpha_i(H_j) F_i$$

$$\alpha_i : \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{n+1} \end{pmatrix} \longmapsto a_i - a_{i+1}$$

$$[E_i, F_j] = \delta_{ij} H_i$$

$$E_i^2 E_{i\pm1} - 2E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0$$

$$F_i^2 F_{i\pm1} - 2F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0$$

Ex. $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, $q \in \mathbb{R}_+^\times$, $q \neq 1$

$U_q(\mathfrak{g})$ is generated by elements E_i, F_i, H_i,

$$E_i = \begin{pmatrix} 0 & & & \\ & \ddots & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad F_i = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & 1 & \ddots & \\ & & & 0 \end{pmatrix}, \quad H_i = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

with the Chevalley-Serre relations:

$$[H_j, E_i] = \alpha_i(H_j) E_i \qquad \alpha_i : \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{n+1} \end{pmatrix} \mapsto a_i - a_{i+1}$$

$$[H_j, F_i] = -\alpha_i(H_j) F_i$$

$$[E_i, F_j] = \delta_{ij} [H_i]_q$$

$$E_i^2 E_{i\pm1} - [2]_q E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0$$

$$F_i^2 F_{i\pm1} - [2]_q F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0$$

where: $[2]_q = q + q^{-1}$, $[H]_q = \frac{q^H - q^{-H}}{q - q^{-1}}$.

Rmk. Of course, one really uses $K_i = q^{H_i}$ instead of H_i as generators.

Irred. representations of a compact semisimple Lie group

K — compact semisimple Lie group (connected, simply connected)

$\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ — complexification of its Lie algebra

Theorem

Irreducible rep'ns of $\mathfrak{g}$ are classified by highest weight $\lambda \in \mathbf{P}^+$.

Explicit structure. . . ?

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Explicit structure. . . ?

- **Weyl character formula (1925):** Action of Cartan subalgebra $\mathfrak{h}$.
- **Gelfand-Tsetlin (1950):** Explicit formulas for action of simple root vectors E_i, F_i, but only for $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{C})$.
- **Pand-Hecht, Wong (1967), & many others:** Then same for $\mathfrak{o}_n$, then $\mathfrak{sp}_n$, then all classical $\mathfrak{g}$.
- **Kashiwara, Lustzig (1990):** Crystal bases (asymptotic formulas + much more)

Example: $V(2\varpi_1 + 2\varpi_2)$

REP$^{\underline{N}}$ OF $U_q(sl_3)$ WITH HIGHEST WEIGHT $(2,0,-2)$

28 September 2011

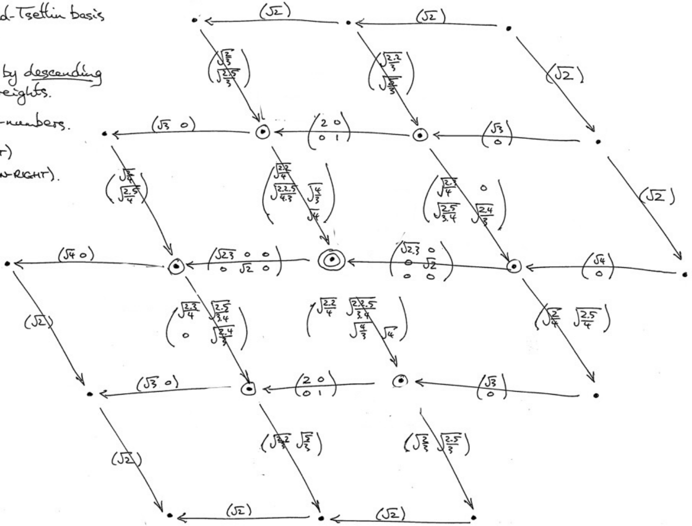
Given w.r.t. orthonormal Gelfand-Tsetlin basis

Notes

- All bases ordered by descending highest $U_q(sl_2)$-weights.
- All numbers are q-numbers.

$\longleftarrow = F_1$ (LEFT)

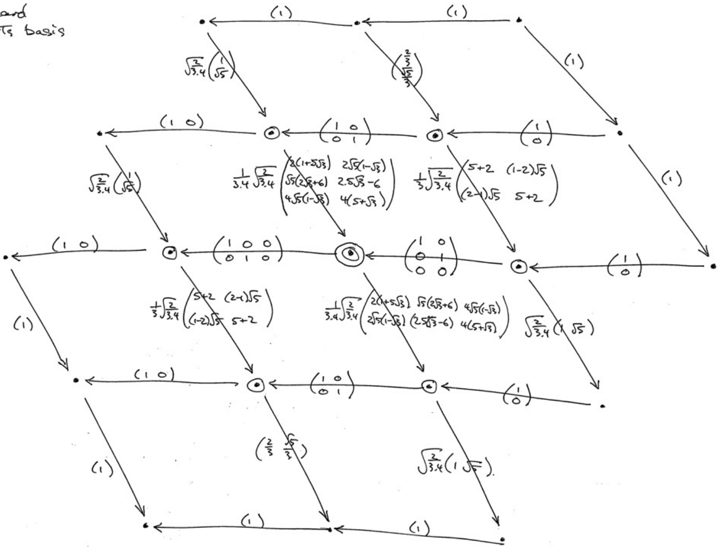
$\longleftarrow = F_2$ (DOWN-RIGHT).



Example: $V(2\varpi_1 + 2\varpi_2)$ — operator phase

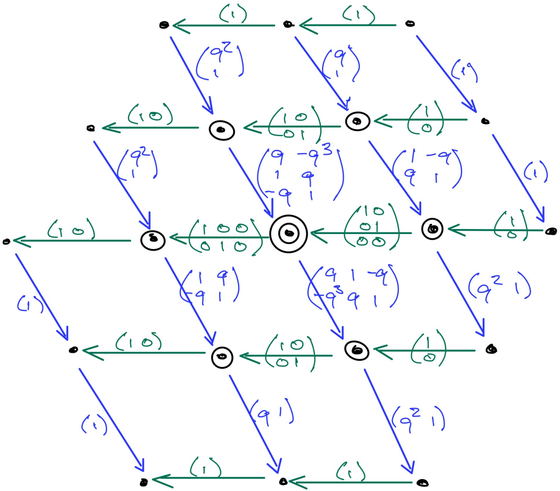
PHASE(F_1) & PHASE(F_2)

Given w.r.t. standard orthonormal G-Ts basis



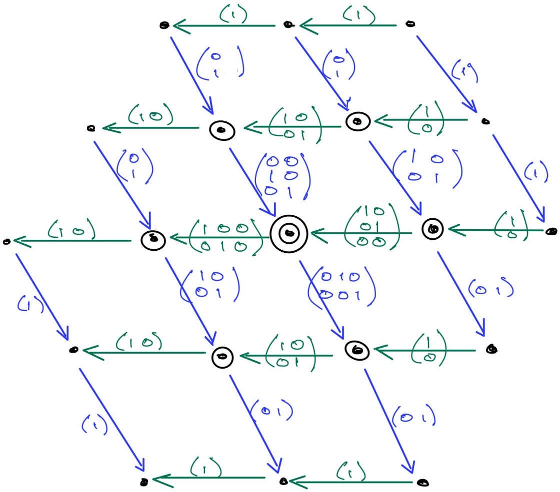
Example: $V(2\varpi_1 + 2\varpi_2)$

DOMINANT TERMS
IN $PH(F_1)$ & $PH(F_2)$
AS $q \to 0$



Example: $V(2\varpi_1 + 2\varpi_2)$

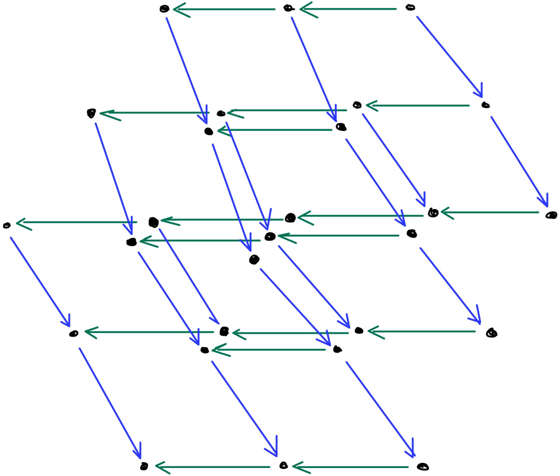
LIMIT AS $q \to 0$



Example: $V(2\varpi_1 + 2\varpi_2)$

LIMIT AS $q \to 0$

(CRYSTAL GRAPH)



Tensor product of crystals

The crystal limit ($q \to 0$) simplifies not just the action of the generators, but also the Clebsch-Gordan coefficients, branching rules, ...

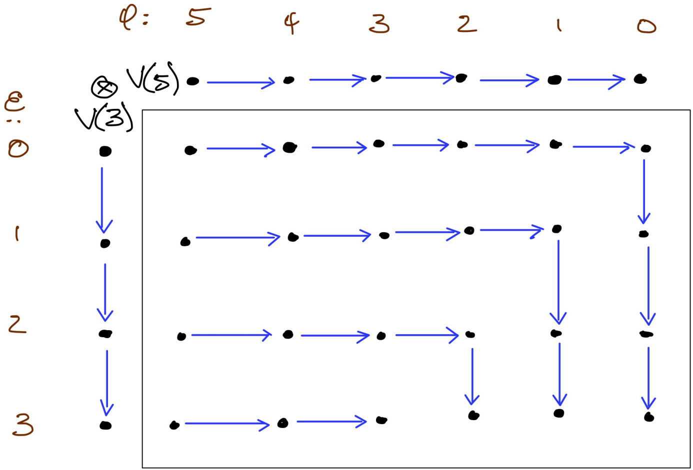
Theorem (Tensor product rule)

If $(\mathcal{L}, \mathcal{B})$, $(\mathcal{L}', \mathcal{B}')$ *are crystal bases for* V, V', *then* $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \times \mathcal{B}')$ *is a crystal basis for* $V \otimes V'$, *with action*

$$\tilde{f}_i : b \otimes c \mapsto \begin{cases} (\tilde{f}_i b) \otimes c, & \text{if } \varphi_i(b) > \varepsilon_i(c) \\ b \otimes (\tilde{f}_i c), & \text{if } \varphi_i(b) \leqslant \varepsilon_i(c) \end{cases}$$

where $\varepsilon_i(b) = \max\{n \mid \tilde{e}_i^n b \neq 0\}$ *and* $\varphi_i(b) = \max\{n \mid \tilde{f}_i^n b \neq 0\}$.

Example: Tensor product of $\mathfrak{sl}_2$ representations



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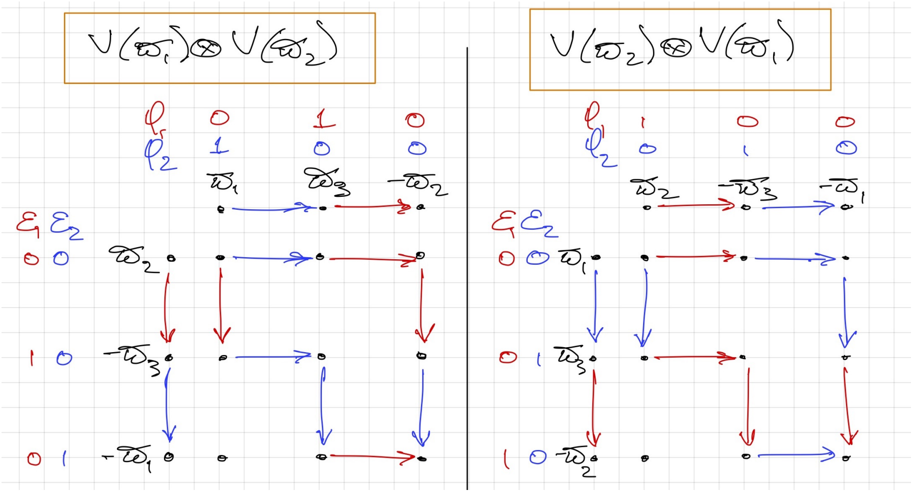
Example: Tensor product of $\mathfrak{sl}_3$ representations



Example: Tensor product of $\mathfrak{sl}_4$ representations